ON THE POWER EFFICIENCY OF A t-TEST FORMED BY PAIRING SAMPLE VALUES

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1. Introduction. Consider two equal sized samples, one from a normal population with mean μ and the other from a normal population with mean ν. Let $x_1, \cdots, x_n$ be the sample values from the population with mean μ and $y_1, \cdots, y_n$ the values from the population with mean ν. If the two populations have the same variance and the two samples are independent, the most powerful tests for comparing μ and ν using these samples (one-sided and symmetrical two-sided) are based on the statistic

$$t_2 = \frac{[\bar{x} - \bar{y} - (\mu - \nu)]\sqrt{n(n-1)}}{\sqrt{\sum_1^n (x_i - \bar{x})^2 + \sum_1^n (y_i - \bar{y})^2}},$$

which has a Student t-distribution with $2n - 2$ degrees of freedom. Tests based on t_2 also have the desirable property of being invariant under permutation of the data in each sample.

Sometimes, however, it is useful to combine the sample values in the form

$$z_i = (x_i - y_i), \qquad (i = 1, \cdots, n).$$

Examples:

(a). When the samples are independent but it is not known that the two populations have the same variance (Behrens-Fisher problem).

(b). When there may be correlation between x_i and y_i, $(i = 1, \cdots, n)$, this correlation being the same for each value of i (i.e. x_i is independent of y_j if $i \neq j$ while each pair x_i, y_i, $(i = 1, \cdots, n)$, has the same normal bivariate distribution).

In both (a) and (b) the z_i are independently normally distributed with the same variance and mean $\mu - \nu$.

The Student t-test for comparing μ and ν using the z_i is based on the statistic

$$t_1 = \frac{[\bar{z} - (\mu - \nu)]\sqrt{n(n-1)}}{\sqrt{\sum_1^n (z_i - \bar{z})^2}} = \frac{[\bar{x} - \bar{y} - (\mu - \nu)]\sqrt{n(n-1)}}{\sqrt{\sum_1^n [x_i - y_i - (\bar{x} - \bar{y})]^2}},$$

which has a Student t-distribution with $n - 1$ degrees of freedom. These tests are not invariant under permutation of the data in each sample.

If it is true that all the sample values are independently distributed with the same variance σ^2, efficiency will be lost by using the test based on t_1 instead of the most powerful test based on t_2. The purpose of this note is to determine the power efficiency of the tests based on t_1 as compared with the corresponding tests based on t_2 for this case.