NOTES

This section is devoted to brief research and expository articles and other short items.

CONVERGENCE OF DISTRIBUTIONS

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Let $f_n(x)$ $(n = 0, 1, 2, \cdots)$ be frequency functions

$$(1) \qquad f_n(x) \geq 0, \qquad \int_{-\infty}^{\infty} f_n(x)\, dx = 1.$$

There are various ways in which the sequence of distributions corresponding to the $f_n(x)$ $(n = 1, 2, \cdots)$ may be said to converge to the distribution corresponding to $f_0(x)$. The definition customarily adopted in mathematical statistics (see e.g. [1]) is equivalent to the condition

$$(a) \qquad \lim_{n\to\infty} \int_{-\infty}^{\xi} f_n(x)\, dx = \int_{-\infty}^{\xi} f_0(x)\, dx \qquad \text{for every } \xi.^1$$

We shall also consider the two further conditions

$$(b) \qquad \lim_{n\to\infty} \int_S f_n(x)\, dx = \int_S f_0(x)\, dx \qquad \text{for every Borel set } S,$$

and

$$(c) \qquad \lim_{n\to\infty} \int_S f_n(x)\, dx = \int_S f_0(x)\, dx \qquad \text{uniformly for all Borel sets } S.$$

It is clear that (c) implies (b) and that (b) implies (a). That the converse implications do not hold is shown by the following examples.

Example 1. Let $f_0(x) = 1$ for $0 \leq x \leq 1$ and 0 elsewhere. Choose and fix any $0 < \epsilon < 1$, set $\delta_n = \epsilon/n \cdot 2^n$, and for $n = 1, 2, \cdots$ let $f_n(x) = 1/n \cdot \delta_n$ for $i/n - \delta_n \leq x \leq i/n$ $(i = 1, 2, \cdots, n)$ and 0 elsewhere. If we denote by S_n the set of all x for which $f_n(x) > 0$ it is easy to see that for $n = 1, 2, \cdots$

$$(2) \qquad 0 \leq \int_{-\infty}^{\xi} f_0(x)\, dx - \int_{-\infty}^{\xi} f_n(x)\, dx < 1/n \qquad \text{for every } \xi,$$

$$(3) \qquad \int_{S_n} f_0(x)\, dx = \epsilon/2^n, \qquad \int_{S_n} f_n(x)\, dx = 1.$$

[1] From a well kown theorem of Pólya the convergence is then necessarily uniform for all ξ.