INDEPENDENCE OF NON-NEGATIVE QUADRATIC FORMS IN NORMALLY CORRELATED VARIABLES

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In a recent paper by the author [5] the following theorem has been mentioned without proof. Though the theorem is very simple and easy to prove the author has not found it elsewhere in the literature.

Theorem. *If two non-negative quadratic forms in normally correlated variables with zero means are uncorrelated the two forms are independent.*

To prove the theorem, let the two forms be

$$(1) \qquad Q_1 = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j, \qquad Q_2 = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} x_i x_j,$$

where the x_i's are normally correlated and all have mean 0. By a well-known theorem on quadratic forms we can reduce Q_1 and Q_2 to the forms

$$(2) \qquad Q_1 = \sum_{i=1}^{n} c_i y_i^2, \qquad Q_2 = \sum_{i=1}^{n} d_i z_i^2,$$

where the y_i's and z_i's are linear functions of the x_i's. In the $2n$-dimensional normal distribution of the y_i's and the z_i's, let ρ_{ij} be the covariance of y_i and z_j. It is then easily shown that the covariance of y_i^2 and z_j^2 is $2\rho_{ij}^2$, and hence that

$$(3) \qquad \text{cov}\,(Q_1, Q_2) = 2 \sum_{i=1}^{n} \sum_{j=1}^{n} c_i d_j \rho_{ij}^2.$$

As the forms are supposed to be non-negative all coefficients in (2) are non-negative. If Q_1 and Q_2 are uncorrelated, each term on the right hand of (3) must vanish. Consequently, if $c_i \neq 0$ and $d_j \neq 0$, we must have $\rho_{ij} = 0$. This means that all y_i's in Q_1 with non-zero coefficients are independent of all z_i's in Q_2 with non-zero coefficients. Hence Q_1 and Q_2 are independent. Q.E.D.

To see if Q_1 and Q_2 are uncorrelated we need an expression for the covariance of the two forms in terms of the coefficients in (1) and the variances and covariances of the original variables x_i. Let A and B be the matrices of the two forms (1). Clearly we may suppose A and B to be symmetric. Let the variance-covariance matrix of the x_i's be L. By straightforward calculations we find

$$(4) \qquad \text{cov}\,(Q_1, Q_2) = 2\, Tr\, ALBL.$$

Here we have used $Tr\, M$ to denote the "trace," i.e. the sum of the diagonal elements in a square matrix M. In case of independent variables with variance 1, we get

$$(5) \qquad \text{cov}\,(Q_1, Q_2) = 2\, Tr\, AB.$$

The formulae (4) and (5) are given in [5].