ON DISTINCT HYPOTHESES

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1. Introduction. The following problem was suggested to one of the authors by Professor Neyman:

Let $X = (X_1, X_4, \cdots, X_n)$ be a chance vector and let h denote any simple hypothesis specifying its distribution. Let H_i be the composite hypothesis that some element h of a set of simple hypotheses $\{h\}_i$, $(i = 0, 1)$, is true, and assume that H_0 and H_1 are known to be exhaustive. Let h_i denote an element of $\{h\}_i$ $(i = 0, 1)$.

For any region W of the sample space S, let $P(W \mid h)$ be the probability that the sample point falls in W when h is true.

We shall call H_0 and H_1 *distinct,* if a region W exists for which

$$P(W \mid h_0) \neq P(W \mid h_1), \qquad \begin{matrix} \text{for all } h_0 \in \{h\}_0 \\ \text{and all } h_1 \in \{h\}_1 . \end{matrix}$$

The problem is to establish necessary and sufficient conditions for two composite hypotheses H_0 and H_1 to be distinct.

For any critical region W for testing H_0 against H_1, let $\gamma(W \mid h)$ be the probability of a wrong decision when h is true, i.e.

$$\gamma(W \mid h) = \begin{cases} P(W \mid h) & \text{for} \quad h \in H_0 \\ 1 - P(W \mid h) & \text{for} \quad h \in H_1 . \end{cases}$$

Suppose now that H_0 and H_1 are not distinct. Then to any W a pair h_0', h_1' exist such that

$$P(W \mid h_0') = P(W \mid h_1'),$$

thus

$$\gamma(W \mid h_0') = 1 - \gamma(W \mid h_1'),$$

and therefore

$$\text{l.u.b. } \gamma(W \mid h) \geq \tfrac{1}{2} \text{ for any } W. \tag{1.1}$$

This property of non-distinct hypotheses leads us to investigate the conditions under which 2 hypotheses allow a test where the maximum probability of a wrong decision is $< \frac{1}{2}$.

The result, in turn, will enable us to state, for an important class of hypotheses a necessary and sufficient condition for 2 composite hypotheses to be distinct.

2. A lemma. We shall now prove the following lemma:

Lemma 2.1. *Assume that X has a density function $p(x)$ and let $H_i = h_i$ be the simple hypothesis that $p(x) = p_i(x)$, $(i = 0, 1)$. Assume that the set R of x's*