ON A THEOREM OF HSU AND ROBBINS

By P. Erdös

Syracuse University

Let $f_1(x), f_2(x), \cdots$ be an infinite sequence of measurable functions defined on a measure space X with measure m, $m(X) = 1$, all having the same distribution function $G(t) = m(x; f_k(x) \le t)$. In a recent paper Hsu and Robbins[1] prove the following theorem: *Assume that*

$$\int_{-\infty}^{\infty} t\, dG(t) = 0, \tag{1}$$

$$\int_{-\infty}^{\infty} t^2\, dG(t) < \infty. \tag{2}$$

Denote by S_n *the set* $\left(x; \left| \sum_{k=1}^{n} f_k(x) \right| > n\right)$, *and put* $M_n = m(S_n)$. *Then* $\sum_{n=1}^{\infty} M_n$ *converges.*

It is clear that the same holds if $\left| \sum_{k=1}^{n} f_k(x) \right| > n$ is replaced by $\left| \sum_{k=1}^{n} f_k(x) \right| > c \cdot n$ (replace $f_k(x)$ by $c \cdot f_k(x)$).

It was conjectured that the conditions (1) and (2) are necessary for the convergence of $\sum_{n=1}^{\infty} M_n$. Dr. Chung pointed it out to me that in this form the conjecture is inaccurate; to see this it suffices to put $f_k(x) = \frac{1}{4}(1 + r_k(x))$ where $r_k(x)$ is the kth Rademacher function. Clearly $|f_k(x)| < 1$; thus $M_n = 0$, thus $\sum_{n=1}^{\infty} M_n$ converges, but $\int_{-\infty}^{\infty} t\, dG(t) \neq 0$. On the other hand we shall show in the present note that the conjecture of Hsu and Robbins is essentially correct. In fact we prove

Theorem I. *The necessary and sufficient condition for the convergence of* $\sum_{n=1}^{\infty} M_n$ *is that*

$$\left| \int_{-\infty}^{\infty} t\, dG(t) \right| < 1, \tag{1'}$$

and (2) *should hold.*

In proving the sufficiency of Theorem I, we can assume without loss of generality that (1) holds. It suffices to replace $f_k(x)$ by $(f_k(x) - C)$ where $C = \int_{-\infty}^{\infty} t dG(t)$. The following proof of the sufficiency of Theorem I (in other words essentially for the theorem of Hsu and Robbins) is simpler and quite different from theirs. Put

$$a_i = m(x; |f_k(x)| > 2^i), \tag{3}$$

[1] *Proc. Nat. Acad. Sciences*, 1947, pp. 25–31.