ON A THEOREM OF HSU AND ROBBINS

BY P. ERDŐS

Syracuse University

Let \( f_1(x), f_2(x), \ldots \) be an infinite sequence of measurable functions defined on a measure space \( X \) with measure \( m, m(X) = 1 \), all having the same distribution function \( G(t) = m(x; f_k(x) \leq t) \). In a recent paper Hsu and Robbins' prove the following theorem: Assume that

\[
(1) \quad \int_{-\infty}^{\infty} t \, dG(t) = 0,
\]

\[
(2) \quad \int_{-\infty}^{\infty} t^2 \, dG(t) < \infty.
\]

Denote by \( S_n \) the set \( \left( x; \sum_{k=1}^{n} f_k(x) > n \right) \), and put \( M_n = m(S_n) \). Then \( \sum_{n=1}^{\infty} M_n \) converges.

It is clear that the same holds if \( \sum_{k=1}^{n} f_k(x) > n \) is replaced by \( \sum_{k=1}^{n} f_k(x) > c \cdot n \) (replace \( f_k(x) \) by \( c \cdot f_k(x) \)).

It was conjectured that the conditions (1) and (2) are necessary for the convergence of \( \sum_{n=1}^{\infty} M_n \). Dr. Chung pointed it out to me that in this form the conjecture is inaccurate; to see this it suffices to put \( f_k(x) = \frac{1}{4}(1 + r_k(x)) \) where \( r_k(x) \) is the \( k \)th Rademacher function. Clearly \( |f_k(x)| < 1 \); thus \( M_n = 0 \), thus \( \sum_{n=1}^{\infty} M_n \) converges, but \( \int_{-\infty}^{\infty} t \, dG(t) \neq 0 \). On the other hand we shall show in the present note that the conjecture of Hsu and Robbins is essentially correct. In fact we prove

**Theorem I.** The necessary and sufficient condition for the convergence of \( \sum_{n=1}^{\infty} M_n \) is that

\[
(1') \quad \int_{-\infty}^{\infty} t \, dG(t) < 1,
\]

and (2) should hold.

In proving the sufficiency of Theorem I, we can assume without loss of generality that (1) holds. It suffices to replace \( f_k(x) \) by \( (f_k(x) - C) \) where \( C = \int_{-\infty}^{\infty} t \, dG(t) \).

The following proof of the sufficiency of Theorem I (in other words essentially for the theorem of Hsu and Robbins) is simpler and quite different from theirs. Put

\[
(3) \quad a_i = m(x; |f_k(x)| > 2^i),
\]


286