ON THE CONVERGENCE OF THE CLASSICAL ITERATIVE METHOD OF SOLVING LINEAR SIMULTANEOUS EQUATIONS[1]

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The classical iterative method, or Seidel method, is a scheme for solving the system of linear algebraic equations

$$\sum_{j=1}^{n} A_{ij} x_j = b_i, \qquad (i = 1, 2, \cdots, n),$$

by successive approximation, as follows:
If $x^{(\nu)} = (x_1^{(\nu)}, x_2^{(\nu)}, \cdots, x_n^{(\nu)})$ is the νth approximation of the solution, the $(\nu + 1)$st approximation, $x^{(\nu+1)} = (x_1^{(\nu+1)}, x_2^{(\nu+1)}, \cdots, x_n^{(\nu+1)})$, is obtained from the relations

$$\begin{cases} A_{11}x_1^{(\nu+1)} + A_{12}x_2^{(\nu)} + A_{13}x_3^{(\nu)} + \cdots + A_{1n}x_n^{(\nu)} = b_1 , \\ A_{21}x_1^{(\nu+1)} + A_{22}x_2^{(\nu+1)} + A_{23}x_3^{(\nu)} + \cdots + A_{2n}x_n^{(\nu)} = b_2 , \\ A_{31}x_1^{(\nu+1)} + A_{32}x_2^{(\nu+1)} + A_{33}x_3^{(\nu+1)} + \cdots + A_{3n}x_n^{(\nu)} = b_3 , \\ \cdots \\ A_{n1}x_1^{(\nu+1)} + A_{n2}x_2^{(\nu+1)} + A_{n3}x_3^{(\nu+1)} + \cdots + A_{nn}x_n^{(\nu+1)} = b_n , \end{cases}$$

$x_1^{(\nu+1)}$ being obtained from the first equation, then $x_2^{(\nu+1)}$ from the second, and so on.

The given system can be written in matrix notation as $Ax = b$ where A is a non-singular square matrix of order n, and x and b are column vectors of order n. Let us define square matrices A_1 and A_2 as follows:

$$(A_1)_{ij} = \begin{cases} A_{ij} \text{ if } i \geq j \\ 0 \text{ if } i < j \end{cases},$$

$$(A_2)_{ij} = \begin{cases} A_{ij} \text{ if } i < j \\ 0 \text{ if } i \geq j \end{cases},$$

(Note that $A_1 + A_2 = A$.).

With this notation the Seidel method can be written as the matric difference equation

$$A_1 x^{(\nu+1)} + A_2 x^{(\nu)} = b.$$

Now various writers, among them C. E. Berry in this journal, (See list of refer-

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