SMOOTHEST APPROXIMATION FORMULAS

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Introduction. Consider a process of approximation which operates on a function $x = x(t)$. The error in the process may be thought of as a sum $R + \delta A$, where R is the error that would be present if x were exact and δA is the error due to errors in x. (Precise definitions are given below.) Suppose that one wishes to choose one process A from a class $\mathfrak{A}$ of processes. In some situations it is appropriate to base the choice on R alone[2]; in others it is appropriate to consider δA.

The primary purpose of the present note is to formulate a criterion of smoothest approximation: That A in $\mathfrak{A}$ is smoothest which minimizes the variance of δA. A criterion based on both R and δA is also suggested. (Sections 1 and 2.) Smoothest approximate integration formulas of one type are derived in Section 3.

Progress in the technique of estimating the covariance function of the errors in x will lead to further applications of the criterion of smoothest approximation.

1. Approximation of a functional. Suppose that X is a space of functions $x = x(t)$ each of which is continuous on $a \leqq t \leqq b$. Let $f[x]$ be a functional defined on X; that is, $f[x]$ is a real number defined for each $x \in X$. For example, X might be the space of functions with second derivatives on $[a, b]$ and $f[x]$ might be $x''(u)$, where u is a fixed number in $[a, b]$.

Suppose that $f[x]$ is to be approximated by a Stieltjes integral

$$A = \int_a^b x(t)\, d\alpha(t), \qquad x \in X, \tag{1}$$

where α is a function of bounded variation. The remainder in the approximation of $f[x]$ by A is

$$R = A - f[x].$$

If the approximation (1) operates on $x + \delta x$ instead of x, the result is $A + \delta A = \int_{t=a}^b (x + \delta x)\, d\alpha$; and the error in the approximation of $f[x]$ by $A + \delta A$ is $R + \delta A$, where

$$\delta A = \int_a^b \delta x(t)\, d\alpha(t). \tag{2}$$

Consider a class $\mathfrak{A}$ of approximations A, each of the form (1). We shall propose a criterion for characterizing the "smoothest A" in $\mathfrak{A}$, relative to the covariance function of the errors δx.

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[2] "Best approximate integration formulas; best approximation formulas," *Amer. Jour. of Math.*, Vol. 71 (1949), pp. 80–91.