A SECOND FORMULA FOR THE PARTIAL SUM OF HYPERGEOMETRIC SERIES HAVING UNITY AS THE FOURTH ARGUMENT

By Hermann von Schelling

Naval Medical Research Laboratory, New London, Connecticut[1]

A convergent hypergeometric series with 1 as fourth argument has been expressed by Gauss, using gamma functions, as follows:

$$(1)\quad F(\alpha, \beta, \gamma; 1) = 1 + \frac{\alpha \cdot \beta}{\gamma \cdot 1} + \frac{\alpha(\alpha + 1) \cdot \beta(\beta + 1)}{\gamma(\gamma + 1) \cdot 1 \cdot 2} + \cdots = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}.$$

Let us denote the νth partial sum of $F(\alpha, \beta, \gamma; 1)$ by $F_\nu(\alpha, \beta, \gamma; 1)$, and let us put

$$(2)\qquad \frac{F_\nu(\alpha, \beta, \gamma; 1)}{F(\alpha, \beta, \gamma; 1)} = G_\nu(\alpha, \beta, \gamma).$$

The following equation is obvious:

$$(3)\qquad G_\nu(\alpha, \beta, \gamma) = G_\nu(\beta, \alpha, \gamma).$$

In [1] it is shown that

$$(4)\qquad G_\nu(\alpha, \beta, \gamma) = 1 - G_\alpha(\nu, \gamma - \beta - \alpha, \gamma - \alpha + \nu)$$

is valid if α is a positive integer.

If $(\gamma - \beta - \alpha)$ is a positive integer, (3) and (4) yield

$$\begin{aligned} G_\nu(\alpha, \beta, \gamma) &= 1 - G_\alpha(\gamma - \beta - \alpha, \nu, \gamma - \alpha + \nu) \\ &= G_{\gamma-\beta-\alpha}(\alpha, \beta, \alpha + \beta + \nu). \end{aligned}$$

In terms of partial sums of the hypergeometric series this becomes

$$(5)\qquad \begin{aligned} &\frac{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}{\Gamma(\gamma)\Gamma(\gamma - \beta - \alpha)} F_\nu(\alpha, \beta, \gamma; 1) \\ &\qquad = \frac{\Gamma(\alpha + \nu)\Gamma(\beta + \nu)}{\Gamma(\nu)\Gamma(\alpha + \beta + \nu)} F_{\gamma-\beta-\alpha}(\alpha, \beta, \alpha + \beta + \nu; 1), \end{aligned}$$

which is a new formula involving partial sums of hypergeometric series with 1 as fourth argument. It is more useful than (4) if $\gamma - \beta - \alpha < \alpha$ or $\gamma < 2\alpha + \beta$.

It is of theoretic interest that the arguments of the new series do not depend on the third argument γ of the original series. Therefore it is possible to develop a simple recursion formula. If we write (5) for $(\gamma - 1)$ instead of γ, the series of the second member has one term less. Subtracting these equations yields after some simplifications

[1] Opinions or conclusions contained in this paper are those of the author. They are not to be construed as necessarily reflecting the views or endorsement of the Navy Department.