FUNDAMENTAL LIMIT THEOREMS OF PROBABILITY THEORY[1]

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no sooner is Proteus caught
than he changes his shape

1. Introduction. The fundamental limit theorems of Probability theory may be classified into two groups. One group deals with the problem of *limit laws* of sequences of sums of random variables, the other deals with the problem of *limits of random variables*, in the sense of almost sure convergence, of such sequences. These problems will be labelled, respectively, the Central Limit Problem (CLP) and the Strong Central Limit Problem (SCLP). Like all mathematical problems, the CLP and SCLP are not static; as answers to old queries are discovered they experience the usual development and new problems arise. The development consists in (i) simplifying proofs and forging general tools out of the special ones (ii) sharpening and strengthening results (iii) finding general notions behind the results obtained and extending their domains of validity. *Analysis of this growth will put in relief the role and the interconnections of the fundamental limit theorems.*

Summary. The growth of the CLP for independent summands can be divided into three (overlapping) periods. The first covers the Bernoulli case and the corresponding limit theorems of Bernoulli, de Moivre and Poisson. The first two theorems gave rise to the notions—from which the classical CLP stems—of the Law of Large Numbers (LLN) and of Normal Convergence (NC). Poisson's approach belongs to the set-up of the modern CLP.

The second period extends over two centuries and is devoted to the extension of the domains of validity of LLN and NC. This is the classical CLP period. Lyapunov's crucial work, submitted to the above treatment, led to the discovery of the natural boundaries of these domains by Lindeberg, Kolmogorov, Feller and P. Lévy.

However, the LLN and NC problems are but two particular cases of the general problem of limit laws of sequences of sums of independent random variables. The coming into sight and the solution of this problem—the third period of the CLP—covers less than ten years. The tools forged for the classical CLP proved to be powerful enough and the final solution is due to P. Lévy, Khintchine, Gnedenko and Doeblin.

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[1] This paper was presented to the New York meeting of the Institute of Mathematical Statistics on December 27, 1949.

Editor's Note: The Institute of Mathematical Statistics has formed a Committee on Special Invited Papers to invite lecturers to deliver expository addresses to the Institute with the understanding that the Special Invited Papers are to be published in the *Annals of Mathematical Statistics*. This paper is the first one invited by the Committee.

[2] This work is supported in part by the Office of Naval Research.