

**INDEPENDENCE OF QUADRATIC FORMS IN NORMALLY
CORRELATED VARIABLES¹**

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The problem to give a necessary and sufficient condition that two quadratic forms in normally correlated variables are independent was treated by many authors [1], [2], [3], [4], [5]. We shall give here also a solution of this problem, which may be a generalization of that given by B. Matérn [6] for nonnegative quadratic forms to the general case.

THEOREM 1. *If two quadratic forms*

$$(1) \quad Q_1 = \sum_{i,j=1}^n a_{ij} x_i x_j, \quad Q_2 = \sum_{i,j=1}^n b_{ij} x_i x_j$$

in normally correlated variables x_1, \dots, x_n with zero means and with the variance matrix I satisfy the following four conditions

$$(2) \quad F_{ij} = E(Q_1^i Q_2^j) - E(Q_1^i) E(Q_2^j) = 0 \quad (i, j = 1, 2),$$

then the relation

$$(3) \quad AB = 0 \quad (A = (a_{ij}), B = (b_{ij}))$$

holds.

COROLLARY 1. *If Q_1, Q_2 in (1) satisfy the four conditions (2), then Q_1 and Q_2 are independent.*

COROLLARY 2. (Necessity portion of the theorem of Craig) *A necessary condition for the independence of Q_1 and Q_2 is $AB = 0$. (The sufficiency was proved by Craig.)*

PROOF OF THEOREM 1. The proof is very simple. Using the values $E(x_k^i) = 0$, ($i = 1, 3, 5, 7$), $E(x_k^2) = 1$, $E(x_k^4) = 3$, $E(x_k^6) = 15$, $E(x_k^8) = 105$ ($k = 1, \dots, n$), we have by a straightforward calculation² the following relations

$$(4) \quad F_{11} = 2Tr(AB),$$

$$(5) \quad F_{12} = 8Tr(AB^2) + 4Tr(AB)Tr(B),$$

$$(6) \quad F_{21} = 8Tr(A^2B) + 4Tr(AB)Tr(A),$$

$$(7) \quad F_{22} = 32Tr(A^2B^2) + 16Tr((AB)^2) + 16Tr(AB^2)Tr(A) + 16Tr(A^2B)Tr(B) \\ + 8Tr(AB)Tr(A)Tr(B) + 8Tr(AB)^2.$$

¹ Presented at the Chapel Hill meeting of the Institute of Mathematical Statistics and Biometric Society March 18, 1950.

² If we apply an orthogonal transformation on (x_1, \dots, x_n) so that A becomes a diagonal form, the calculation becomes simpler than with the general form. We may note here also the fact that we need not assume that x_1, \dots, x_n are normally correlated, but we use only the values of $E(x_k^i)$ ($i = 1, \dots, 8$) for our proof.