TABLE OF THE ASYMPTOTIC DISTRIBUTION OF THE SECOND EXTREME

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The asymptotic distributions of the extreme values taken from an initial distribution of the exponential type are now widely used, for example in flood control [6] and in problems connected with the breaking strength of material [1]. Therefore, the corresponding distribution of the penultimate (and of the second) value may also be of practical interest.

Let $F(x)$ be the initial probability; let $f(x) = F'(x)$ be the initial density (distribution). Let n be a large sample size; let the rank $m(m << n)$ be counted from the top. Finally, let the parameters u_m and α_m be defined as the solutions of

(1) $$F(u_m) = 1 - m/n; \qquad \alpha_m = nf(u_m)/m.$$

Then the asymptotic distribution $\varphi_m(x_m)$ of the mth largest value x_m is [2]

$$\varphi_m(x_m) = \frac{m^m}{\Gamma(m)} \alpha_m \exp[-my_m - me^{-y_m}],$$

where

$$y_m = \alpha_m(x_m - u_m),$$

provided that the initial distribution is of the exponential type. The asymptotic distribution ${}_m\varphi({}_mx)$ of the mth smallest value is

$${}_m\varphi({}_mx) = \varphi_m(-x_m).$$

The probability function $\Phi_m(x_m)$ is obtained from

$$\Phi_m(x_m) = \frac{m^m}{\Gamma(m)} \int_{-\infty}^{y_m} \exp[-my - me^{-y}]\, dy$$
$$= \frac{1}{\Gamma(m)} \int_{me^{-y_m}}^{\infty} z^{m-1} e^{-z}\, dz,$$

whence

(2) $$\Phi_m(x_m) = 1 - I(t_m, m - 1),$$

where

$$t_m = \sqrt{m}\, e^{-y_m}$$

and I is the incomplete Gamma function ratio of Karl Pearson [5]. In the special case $m = 2$, the probability function of the penultimate value is

(3) $$\Phi_2(x_2) = 1 - I(\sqrt{2}\, e^{-y_2}, 1).$$

The modal penultimate value is, of course, u_2, and the intervals corresponding