NOTES

This section is devoted to brief research and expository articles and other short items.

ON A THEOREM OF LYAPUNOV

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The purpose of this note is to point out two extensions of the following theorem of Lyapunov[1], and to note an interesting statistical consequence of each.[2]

Lyapunov's Theorem: *Let $u_1, \cdots, u_n$ be non-atomic[3] measures on a Borel field $\mathcal{B}$ of subsets of a space X. The set R of vectors $[u_1(E), \cdots, u_n(E)]$, $E \epsilon \mathcal{B}$, is convex, i.e., if $r_1, r_2 \epsilon R$, so does $tr_1 + (1 - t)r_2$ for $0 \leq t \leq 1$.*

Extension 1. *Let $u_1, \cdots, u_n$ be non-atomic measures on a Borel field of subsets of a space X and let A be any subset of n-dimensional Euclidean space. Let $f = a(x) = [a_1(x), \cdots, a_n(x)]$ be any $\mathcal{B}$-measurable function defined on X with values in A, and define $v(f) = [\int a_1(x)\, du_1, \cdots, \int a_n(x)\, du_n]$. The set of vectors $v(f)$ is convex.*

Lyapunov's theorem is the special case in which A consists of two points $(0, \cdots, 0)$ and $(1, \cdots, 1)$.

Proof. Let $v(f_i) = v_i$, $f_i = [a_{i1}(x), \cdots, a_{in}(x)]$, $i = 1, 2$, and consider the $2n$-dimensional measure

$$w(E) = \int_E a_{11}(x)\, du_1 \cdots, \int_E a_{1n}(x)\, du_n, \int_E a_{21}(x)\, du_1, \cdots, \int_E a_{2n}(x)\, du_n.$$

Since $w(N) = (0, \cdots, 0)$ where N is the null set, $w(X) = (v_1, v_2)$, for any t, $0 < t < 1$, there is, by Lyapunov's theorem, a set $E \epsilon \mathcal{B}$ with $w(E) = (tv_1, tv_2)$,

[1] "Sur les fonctions-vecteurs complètement additives," *Bull. Acad. Sci. URSS. Sér. Math.* Vol. 4 (1940), pp. 465–478. For a simplified proof of Lyapunov's results, see Halmos, "The range of a vector measure," *Bull. Amer. Math. Soc.*, Vol. 54 (1948), pp. 416–421.

[2] Since this note was submitted, results obtained earlier by Dvoretzky, Wald, and Wolfowitz have appeared in the April 1950 *Proceedings of the National Academy of Sciences*. Their results are closely related to those presented here, and anticipate the general conclusion reached here: that in dealing with non-atomic distributions, mixed strategies are unnecessary. Their principal tool is also an extension of Lyapunov's theorem; their extension does not appear to contain or be contained in either of the extensions given here. The situation considered here is more general in that an infinite number of possible terminal actions are possible, but more restricted in that only mixtures of a finite number of pure strategies are considered here.

[3] A measure u is non-atomic if every set of non-zero measure has a subset of different non-zero measure.