AN INVERSE MATRIX ADJUSTMENT ARISING IN DISCRIMINANT ANALYSIS

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1. Introduction. The adjustment of an inverse matrix arising from the change of a single element, or of elements in a single row or column, in the original matrix has recently been discussed by Sherman and Morrison [1, 2]. In discriminant function analysis the adjustment due to the addition of a degenerate matrix of rank one to the original matrix has sometimes been required, and the method used by the writer is described in this note. It will be noticed that this case includes the cases considered by Sherman and Morrison.

2. General formula. The new square matrix can always be written in the form

$$\mathbf{B} = \mathbf{A} + \mathbf{u}\mathbf{v}', \tag{1}$$

where $\mathbf{u}$ is a column vector (single column matrix), and $\mathbf{v}'$ a row vector (dashes denote matrix transposes). We write formally

$$\begin{aligned} \mathbf{B}^{-1} &= (\mathbf{A} + \mathbf{u}\mathbf{v}')^{-1} = \mathbf{A}^{-1}(1 + \mathbf{u}\mathbf{v}'\mathbf{A}^{-1})^{-1} \\ &= \mathbf{A}^{-1}(1 - \mathbf{u}\mathbf{v}'\mathbf{A}^{-1} + \mathbf{u}\mathbf{v}'\mathbf{A}^{-1}\mathbf{u}\mathbf{v}'\mathbf{A}^{-1} - \cdots) \\ &= \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{u}\cdot\mathbf{v}'\mathbf{A}^{-1}\{1 - \mathbf{v}'\mathbf{A}^{-1}\mathbf{u} + (\mathbf{v}'\mathbf{A}^{-1}\mathbf{u})^2 - \cdots\} \\ &= \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{u}\cdot\mathbf{v}'\mathbf{A}^{-1}}{1 + \mathbf{v}'\mathbf{A}^{-1}\mathbf{u}}, \end{aligned} \tag{2}$$

which has the same simple structure as (1) and can be determined when $\mathbf{A}^{-1}$ is known. To check this formal result, we may easily verify that pre- or post-multiplication of the expression (2) by $\mathbf{B}$ gives the unit matrix.

3. Numerical example in discriminant analysis. The general regression relation between two sample matrices $\mathbf{S}_2$ and $\mathbf{S}_1$ may be written (Bartlett [3])

$$\mathbf{S}_2 = \mathbf{C}_{21}\mathbf{C}_{11}^{-1}\mathbf{S}_1 + \mathbf{S}_{2.1}\,. \tag{3}$$

Here the n observations of any variable (measured if necessary from the general mean) comprise one row in the appropriate matrix, $\mathbf{S}_2$ and $\mathbf{S}_1$ representing respectively the dependent and independent variables. $\mathbf{S}_2\mathbf{S}_1'$ is written $\mathbf{C}_{21}$ for convenience, and similarly for $\mathbf{C}_{11}$, $\mathbf{C}_{22}$; also $\mathbf{C}_{22.1} = \mathbf{S}_{2.1}\mathbf{S}_{2.1}'$. In discriminant analysis in its strict sense $\mathbf{S}_1$ stands for a single dummy variable serving to isolate a group or other contrast between the proper random variables $\mathbf{S}_2$. In that case the equation

$$\mathbf{C}_{22} = \mathbf{C}_{21}\mathbf{C}_{11}^{-1}\mathbf{C}_{12} + \mathbf{C}_{22.1} \tag{4}$$