LINEAR TRANSFORMATIONS AND THE PRODUCT-MOMENT MATRIX

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Using linear transformations G. Rasch has deduced Wishart's distribution in his paper on "A Functional Equation for Wishart's Distribution" (*Annals of Math. Stat.*, Vol. 19 (1948), pp. 262–266). This note is of the nature of some observations on the Jacobian of the transformation induced by a linear transformation of coordinates with constant coefficients in the distinct elements of the product-moment matrix of a sample of n vectors, each of k components, drawn from a universe of a normal k-variate distribution with zero means. If the r-th vector of the drawn sample has components $(x_i^{(r)})$, $r = 1, 2, \cdots, n$, $i = 1, 2, \cdots, k$, the sum of the products of the i-th and j-th components of each of the n vectors is denoted by

$$M_{ij} = \sum_{r=1}^{n} x_i^{(r)} x_j^{(r)}.$$

Let the variables of the vector variate be $x_1, x_2, \cdots, x_k$, or shortly (x), and let a nonsingular (i.e., reversible) linear transformation of the variables with constant coefficients be made from (x) to (y), viz.,

$$x_r = \sum_{i=1}^{k} a_{ri} y_i \qquad (r = 1, 2, \cdots, k).$$

The distinct elements $M_{11}, M_{12}, \cdots, M_{1k}, M_{22}, M_{23}, \cdots, M_{kk}$ undergo a consequential or induced transformation which is also linear in terms of the corresponding elements of the product-moment matrix $\| M'_{ij} \|$ of the same n-vector sample in the coordinates (y).

Let the matrix of the coefficients of the induced transformation which is also the matrix of partial derivatives in this case be denoted by $\| J \|$, and let its determinant which is the Jacobian of the transformation be denoted by J. The elements of $\| J \|$ are functions of the elements of $\| a_{ri} \|$. When $\| a_{ri} \|$ is in the diagonal form, so is also $\| J \|$ with elements $a_{11}a_{11}, a_{11}a_{22}, \cdots, a_{11}a_{kk}, a_{22}a_{22}, a_{22}a_{33}, \cdots, a_{kk}a_{kk}$. In this case we have $J = A^{k+1}$, where A stands for the determinant of $\| a_{ri} \|$ which is nonzero on account of the nonsingularity of the transformation considered. It is then natural to ask the question whether it can be asserted that the same relationship holds even when the matrix $\| a_{ri} \|$ is not in the diagonal form or reducible to it. The answer is in the affirmative, the result being a particular case of the following theorem of Escherich (see C. C. Macduffee, *Theory of Matrices*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Julius Springer, Berlin, 1933, p. 86, theorem 44.3): *The determinant of the m-th power-matrix*[1] *with a nonvanishing determinant* A *is* $A^{(m+k-1)C_k}$, *where*

[1] It may be noted that the m-th power-matrix mentioned in Escherich's theorem is not the matrix multiplied by itself by the ordinary matrix multiplication. For its definition see Macduffee, loc. cit., p. 85.