ON UNIFORMLY CONSISTENT TESTS

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1. Introduction. If we wish to decide on the true distribution of a random variable known to be distributed according to one or the other of two given distributions F_0 and F_1, then, no matter how small a bound is given in advance, it is always possible to devise a test based on a sufficiently large number of independent observations for which the probabilities of erroneous decisions are smaller than the previously assigned bound. A sequence of tests for which the corresponding probabilities of errors tend to zero has been called consistent [1].[1]

Let us suppose now that all we know about the true distribution of some random variable is that it belongs to one of two given families of distributions and it is desired to decide which of the two it belongs to; i.e., we have to test a composite hypothesis. It may again be possible to construct a sequence of tests $\{T_j\}$, $j = 1, 2, \cdots$, such that for any $\epsilon > 0$ there exists an index N such that for $j > N$ the probabilities of errors corresponding to T_j are smaller than ϵ. The sequence $\{T_j\}$ may then be called uniformly consistent. Conditions under which uniformly consistent tests exist have been given by von Mises [5], and by Wald [2], [3], [4], as implied, for example, by his proof of the uniform consistency of the likelihood ratio test. In this paper a different set of conditions is given which do not restrict in any way the nature of the distribution functions considered. It is also shown that the conditions to be described are satisfied in a large class of cases occurring in practical statistics.

Since the results we are to prove have their counterpart in abstract measure theory we shall take advantage of that method. The reader will have no difficulty in establishing the correspondence between the statistical and measure theoretical formulation.

Notations. Let X be an arbitrary set and $\mathfrak{B}$ a Borel field of subsets B of X. Let $\mathfrak{M}(\mathfrak{B})$ be the family of all probability measures $m(B)$ defined on $\mathfrak{B}$, i.e., the family of all countably additive nonnegative set functions defined on $\mathfrak{B}$ for which $m(X) = 1$. Hereafter a "measure" will denote an element of $\mathfrak{M}(\mathfrak{B})$ and a "set of measures" a subset of $\mathfrak{M}(\mathfrak{B})$. For any positive integer k, let X^k be the kth direct product of X by itself, $\mathfrak{B}^k$ the kth direct product of $\mathfrak{B}$ by itself, $\mathfrak{C}^{\cdot k}$ the field consisting of all finite sums of sets of $\mathfrak{B}^k$ and $\mathfrak{B}^{\cdot k}$ the smallest σ-field containing $\mathfrak{C}^{\cdot k}$. For any measure m on $\mathfrak{B}$, we define m^k in the usual way as the unique measure defined on $\mathfrak{B}^{\cdot k}$ for which

$$m^k\left(\sum_{i=1}^{l} B_{i1}\cdot B_{i2}\cdot \cdots \cdot B_{ik}\right) = \sum_{i=1}^{l} m(B_{i1})m(B_{i2}) \cdots m(B_{ik})$$

for any disjoint system $B_{ij} \varepsilon \mathfrak{B}$, $i = 1, 2, \cdots, l$; $j = 1, 2, \cdots, k$, where l is an arbitrary positive integer.

[1] This is a slightly modified form of the definition in [1].