A PROPERTY OF SOME TESTS OF COMPOSITE HYPOTHESES

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In all common statistical tests, a result significant at the 1 per cent level is necessarily significant at the 5 per cent level. In this note we show that this statement is not true for all statistical tests. More precisely, for any α_1, α_2 satisfying $0 < \alpha_1 < \alpha_2 < 1$, we construct a composite hypothesis H_0 and a simple hypothesis H_1 such that there are sets w_1, w_2 in the sample space which are the unique most powerful critical regions of size α_1, α_2, respectively, for testing H_0 against H_1. Furthermore, w_1, w_2 are similar regions. But w_2 does not contain w_1.

Let X be a random variable which can take one of the four values 1, 2, 3, 4. Let H_0 consist of two simple hypotheses H_0' and H_0'', where H_0' states that $P\{X = i\} = p_i'$, and H_0'' states that $P\{X = i\} = p_i''$; and let H_1 state that $P\{X = i\} = p_i$ for $i = 1, 2, 3, 4$. Later, we shall determine appropriate positive values for the p_i, p_i', p_i''. Let $\pi_i' = p_i'/p_i$, $\pi_i'' = p_i''/p_i$. By a slight modification of the Neyman-Pearson lemma [1] (see also [2]), the region w_1 consisting of the points $x = 1$ and $x = 2$, and the region w_2 consisting of the points $x = 1$ and $x = 3$, are both most powerful critical regions and similar if and only if

$$\text{(a)} \qquad \begin{aligned} p_1\pi_1' + p_2\pi_2' &= p_1\pi_1'' + p_2\pi_2'', \\ p_1\pi_1' + p_3\pi_3' &= p_1\pi_1'' + p_3\pi_3''; \end{aligned}$$

(b) there exist constants a_1, a_2, b_1, $b_2 \geq 0$ with $a_1 + b_1 > 0$, $a_2 + b_2 > 0$, such that $a_1\pi_1' + b_1\pi_1''$, $a_1\pi_2' + b_1\pi_2''$ are both less than or equal to $a_1\pi_3' + b_1\pi_3''$, $a_1\pi_4' + b_1\pi_4''$, and $a_2\pi_1' + b_2\pi_1''$, $a_2\pi_3' + b_2\pi_3''$ are both less than or equal to $a_2\pi_2' + b_2\pi_2''$, $a_2\pi_4' + b_2\pi_4''$. Expressed geometrically in the (π', π'')-plane, if a_1, a_2, b_1, $b_2 > 0$ and "less than" holds in all the above relations (which will always be the case in our construction), this means that the line joining points 2 and 3 intersects both axes at positive values, and the point 1 is inside and point 4 outside the triangle formed by this line and the coordinate axes. Of course H_0', H_0'', H_1 are all probability distributions and all of the points 1, 2, 3, 4 are to have positive probabilities, so that we want

$$\text{(c)} \qquad \sum p_i = 1, \qquad \sum p_i\pi_i' = 1, \qquad \sum p_i\pi_i'' = 1;$$

$$\text{(d)} \qquad p_i > 0, \qquad \pi_i' > 0, \qquad \pi_i'' > 0.$$

We shall show that conditions (a), (b), (c), (d) can be satisfied in a great variety of ways. Choose π_{i0}', π_{i0}'' so that $\pi_{10}'' > \pi_{10}'$, $\pi_{20}' > \pi_{20}''$, $\pi_{30}' > \pi_{30}''$, $\pi_{40}'' > \pi_{40}'$, and (b) is satisfied when π_i', π_i'' are replaced by π_{i0}', π_{i0}'', respectively. Let p_{10} be an arbitrary nonnegative number. Choose p_{20}, $p_{30} \geq 0$ so that

$$\text{(1)} \qquad \begin{aligned} p_{10}\pi_{10}' + p_{20}\pi_{20}' &= p_{10}\pi_{10}'' + p_{20}\pi_{20}'', \\ p_{10}\pi_{10}' + p_{30}\pi_{30}' &= p_{10}\pi_{10}'' + p_{30}\pi_{30}''. \end{aligned}$$

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