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ALMOST SUBMINIMAX AND BIASED MINIMAX PROCEDURES[1]

BY P. FRANK AND J. KIEFER

Columbia University

Robbins [1] emphasized the notion of an "almost subminimax" procedure[2] and gave an example of such a procedure. The examples in this paper have been constructed with a view to simplicity and to the indication of the underlying mechanism which makes subminimax solutions exist in certain decision problems. At the same time we point out another potentially undesirable property of a minimax procedure—biasedness.

All our examples fall within the following framework. A sample of one is taken from a population whose distribution is one of n given distributions: $F_1(x), F_2(x), \cdots, F_n(x)$. There are n decisions: $d_1, \cdots, d_n$. The weight function is $W(F_i, d_j) = 0$ if $i = j$ and $= 1$ otherwise. Instead of a finite number of F's, we may have a sequence of F's with a corresponding sequence of decisions. In all our examples each of the F's will be a uniform distribution over a finite interval of the x-axis, and our decision procedures will be randomized. These restrictions are made only for arithmetical simplicity.

With this setup, the risk when F_i is the true distribution is equal to the probability of not making decision d_i, which we will denote $r(F_i)$. We will not give an exact definition of an almost subminimax procedure, but just say that a procedure is almost subminimax if its maximum risk is "a little greater" than that of the minimax procedure (which risk is the same for all minimax procedures in our examples) and on the other hand its risk is "a lot less" than that of the minimax for "most of " the F's. Our examples will conform with this "definition" for almost any reasonable interpretation of the phrases in the quotes.

The first example will give an indication of the mechanism which makes a subminimax example possible. Let $F_1(x)$ be the uniform distribution on the interval $1 - a$ to 1, where $a > 0$ and small. Let $F_2(x)$ be the uniform distribution on the interval 0 to 1. An admissible minimax procedure to decide between d_1

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[2] The examples of this paper fall into the framework of the definition in [1] of an "asymptotically subminimax solution" if each example is replaced by a sequence of examples whose a's approach zero. The present nomenclature was suggested as more suitable here.