DISTRIBUTION OF THE ORDINAL NUMBER OF SIMULTANEOUS EVENTS WHICH LAST DURING A FINITE TIME[1]

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1. Introduction. The probability of drawing a white ball from an urn is p, and the complementary probability of getting a black ball is $(1 - p) = q$. One ball is drawn and returned during one time unit. When a white ball appears, the play is interrupted for k time units. Then it starts anew.

If it happens that at the nth time unit a white ball occurs, we ask for the probability $w(m; n, k, p)$ that it is the mth ball since the first beginning of the trials. We are interested in the mean $E(m)$ and in the variance $\text{Var}(m)$, and in simple approximations for $E(m/n)$ and $\text{Var}(m/n)$ when n is large.

2. The probability distribution. Let us start with the relative probabilities. If the first white ball appeared at the nth moment, $(n - 1)$ black balls preceded, which means that the relative probability is q^{n-1}. If it was the second white ball, the number of black balls was reduced by $(k + 1)$, k for one interruption of the play lasting k time units and 1 for the first white ball, which occurred with the probability p. The group of $[(n - 1) - (k + 1)]$ black balls may be broken into any two parts, including the case of one being empty. That makes $\binom{(n-1)-k}{1}$ possibilities. Therefore the relative probability for $m = 2$ is

$$q^{n-1}\binom{(n-1)-k(m-1)}{m-1}\left(\frac{p}{q^{k+1}}\right)^{m-1}, \tag{1}$$

with $m = 2$. It is easy to verify step by step that the general formula is correct for $m = 1, 2, \cdots, 1 + \left[\frac{n-1}{k+1}\right]$. Hence the preliminary answer to our problem is

$$w(m; n, k, p) = \frac{1}{C}\, q^{n-1}\binom{(n-1)-k(m-1)}{m-1}\left(\frac{p}{q^{k+1}}\right)^{m-1}, \quad k = 0, 1, 2, \cdots, \quad m = 1, 2, \cdots, 1 + \left[\frac{n-1}{k+1}\right], \tag{2}$$

where $[a]$ means the largest positive integer $\leqq a$. The constant C has to be determined by

$$\sum_{m=1}^{1+[(n-1)/(k+1)]} w(m; n, k, p) = 1. \tag{3}$$

[1] Opinions or conclusions contained in this paper are those of the author. They are not to be construed as necessarily reflecting the views or endorsement of the Navy Department.