THE EXACT DISTRIBUTION OF THE EXTREMAL QUOTIENT

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0. Problem. The only quotients considered up to now are those of two observations taken from different distributions. Instead of these statistics, we consider the quotient Q of the extremes (henceforth called the *extremal quotient*) for $n \geq 2$ independent observations taken from the same distribution. This quotient of the extremes has sometimes been used by climatologists [3]. Since it is obviously not affected by changes of scale, its use may be of interest in cases where the scale plays no role. The sensitivity of the extremal quotient to changes in origin is brought out by consideration of uniform distributions where the extremal quotient for a nonnegative variate has just the opposite qualities of the extremal quotient taken from a nonpositive variate.

The asymptotic distribution of the extremal quotient was given in a previous paper [1]. However, the exact distribution of this statistic has never been studied before.[1]

1. The distribution. Let $f(x)$ and $F(x)$ be the density and cumulative probability function of a variate X where $-\omega_1 \leq X \leq \omega_2$. Let X_n be the largest and X_1 the smallest value in a sample of size n ($n \geq 2$). Then the extremal quotient is $Q = X_n/X_1$. The exact cumulative probability function $H(q)$ will be given in terms of pseudo probability functions

$$\begin{aligned} \Phi_1(q) &= Pr\{\ \ 1 \leq Q \leq q\}, && X \geq 0, \\ \Phi_2(q) &= Pr\{-1 \leq Q \leq q\}, && X_1 \leq 0, \quad X_n \geq 0, \\ \Phi_3(q) &= Pr\{\ \ 0 \leq Q \leq q\}, && X \leq 0. \end{aligned} \tag{1.1}$$

These would be cumulative probability functions of Q if the extremal quotient were restricted to the quadrant indicated by the subscript (see Figure 1). In general the cumulative probability function is

$$H(q) = \begin{cases} \Phi_2(q), & q \leq 0, \\ \Phi_2(0) + \Phi_3(q), & 0 \leq q \leq 1, \\ \Phi_2(0) + \Phi_3(1) + \Phi_1(q), & q \geq 1. \end{cases} \tag{1.2}$$

Integrating the joint density of the extremes,

$$w(x_1, x_2) = n(n-1)f(x_1)[F(x_n) - F(x_1)]^{n-1}f(x_n), \tag{1.3}$$

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