NOTES

A GENERAL CONCEPT OF UNBIASEDNESS

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The term unbiasedness was introduced by Neyman and Pearson [1] in connection with hypothesis testing. A test of the hypothesis $\theta \varepsilon \omega$ against the alternatives $\theta \varepsilon \Omega - \omega$ is said to be unbiased at level α if its power function β satisfies

$$\beta(\theta) \leq \alpha \text{ for } \theta \varepsilon \omega, \quad \beta(\theta) \geq \alpha \text{ for } \theta \varepsilon \Omega - \omega. \tag{1}$$

In 1937 Neyman [2] developed a theory of estimation by confidence sets. He established a duality with the theory of hypothesis testing, so that to each notion of one theory corresponds an analogous one in the other. In particular, he defined a family of confidence sets $A(x)$ to be unbiased if

$$P_\theta(A(X) \supset \theta') \leq P_\theta(A(X) \supset \theta) \text{ for all } \theta \text{ and } \theta'. \tag{2}$$

While the above two definitions are closely related, a third use of the term unbiasedness was made in a rather different context. In presenting their version of the Gauss-Markov theorem on least squares David and Neyman [3] defined a point estimate $\delta(X)$ of $g(\theta)$ to be unbiased if its expectation coincides with the estimated value, that is, if

$$E_\theta \delta(X) \equiv g(\theta). \tag{3}$$

It was pointed out later by Brown [4] that one obtains other analogous definitions by postulating that some central value of the distribution of $\delta(X)$ other than the mean coincides with the estimated value. Using the median as an example he defined $\delta(X)$ to be median-unbiased if

$$P_\theta(\delta(X) > g(\theta)) = P_\theta(\delta(X) < g(\theta)) \text{ for all } \theta. \tag{4}$$

In view of Wald's theory of decision functions [5] it seems tempting to try to give a definition of unbiasedness at the level of generality of this theory. Suppose we are concerned with a decision problem where the loss resulting from a decision $\delta(X)$ is $W(\theta, \delta(X))$ when the true parameter value is θ. In analogy with (2) we shall say that a decision procedure $\delta(X)$ is unbiased if for each θ

$$E_\theta W(\theta', \delta(X)) = \min \text{ when } \theta' = \theta. \tag{5}$$

This clearly reduces to Neyman's definition for confidence sets if one uses for loss function,

$$W(\theta, \delta(x)) = \begin{cases} 0 \text{ if the confidence set } \delta(x) \text{ covers } \theta, \\ 1 \text{ otherwise.} \end{cases} \tag{6}$$