where $0 < \beta < 1, \lambda > 0$. We find that

$$a_n = \frac{\lambda + n - 1}{n!} \beta(1 + \lambda\beta) \cdots (n - 1 + [\lambda + n - 2]\beta), \qquad n = 1, 2, \cdots,$$

with $a_0 = 1$. In particular, when $\lambda = 1$, we have

$$a_n = (1 + \beta)^{n-1}\beta, \qquad n = 1, 2, \cdots,$$

with $a_0 = 1$, and each row of **A** is a truncated modified geometric distribution.

REFERENCE

[1] WILLIAM FELLER. *An Introduction to Probability Theory and its Applications*, John Wiley and Sons, 1950.

ON MINIMUM VARIANCE ESTIMATORS[1]

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Chapman and Robbins [1] have given a simple improvement on the Cramér-Rao inequality without postulating the regularity assumptions under which the latter is usually proved. The purpose of this note is to show by examples how a similarly derived stronger inequality (see equation (2)) may be used to verify that certain estimators are uniform minimum variance unbiased estimators. This stronger inequality is that which (under additional restrictions) was shown in [2] to be the best possible, but is in a more useful form for applications than the form given in [2]. For simplicity we consider only an inequality on the variance of unbiased estimators, but inequalities on other moments than the second (see [2]), or for biased estimators, may be found similarly. The two examples considered here are ones where the regularity conditions of [2] are not satisfied, where the method of [1] does not give the best bound, and where the method of this note is used to find the best bound and thus to verify that certain estimators are uniform minimum variance unbiased. (For the examples considered this also follows from completeness of the sufficient statistic; the method used here applies, of course, more generally.)

Let X be a chance variable with density $f(x; \theta)$ with respect to some fixed σ-finite measure μ. ($\theta \varepsilon \Omega$, $x \varepsilon \mathfrak{X}$). We suppose suitable Borel fields to be given and $f(x; \theta)$ to be measurable in its arguments. Ω is a subset of the real line. For each θ, let $\Omega_\theta = \{h \mid (\theta + h) \varepsilon \Omega\}$. For fixed θ, let λ_1 and λ_2 be any two probability measures on Ω_θ such that $E_i h = \int_{\Omega_\theta} h d\lambda_i(h)$ exists for $i = 1, 2$. Then, for any

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