NOTES

A MARKOV CHAIN DERIVATION OF DISCRETE DISTRIBUTIONS

By F. G. Foster

Magdalen College, Oxford

Let an irreducible, aperiodic Markov chain[1] have the matrix of transition probabilities, $\mathbf{A} = [p_{ij}]$ $(i, j = 0, 1, 2, \cdots)$. Then as usual we shall have

$$p_{ij} \geqq 0 \qquad \text{for all } i \text{ and } j,$$

$$\sum_{j=0}^{\infty} p_{ij} = 1 \qquad \text{for all } i.$$

It is known ([1], p. 325) that the nth power of $\mathbf{A}$, $\mathbf{A}^n$, tends to a limiting matrix as $n \to \infty$

$$\lim_{n\to\infty} \mathbf{A}^n = \mathbf{B},$$

and $\mathbf{B}$ will either be null or have the identical rows,

$$\mathbf{x} = (x_0, x_1, \cdots),$$

such that $x_i > 0$ for all i and $\sum_{i=0}^{\infty} x_i = 1$. Moreover we shall have

$$\mathbf{xA} = \mathbf{x}.$$

In this way we may make correspond to any matrix $\mathbf{A}$, of the type under consideration, either the null vector or a probability distribution represented by $\mathbf{x}$. Conversely, to any distribution $\mathbf{x}$ there will correspond a matrix $\mathbf{A}$ (not necessarily unique). A method of constructing such a matrix is given below and illustrated with some examples.

Let $\{a_i\}$ $(i = 0, 1, 2, \cdots)$ be a sequence of positive numbers and define $A_n = \sum_{i=0}^{n} a_i$ $(n = 0, 1, 2, \cdots)$. Now let

$$\mathbf{A} = \begin{bmatrix} \frac{a_0}{A_1} & \frac{a_1}{A_1} & 0 & 0 & 0 & \cdots \\ \frac{a_0}{A_2} & \frac{a_1}{A_2} & \frac{a_2}{A_2} & 0 & 0 & \cdots \\ \frac{a_0}{A_3} & \frac{a_1}{A_3} & \frac{a_2}{A_3} & \frac{a_3}{A_3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Then $\mathbf{A}$ satisfies the usual conditions for being a transition probability matrix;

[1] For definitions of all terms used see [1].