Selected Techniques of Statistical Analysis, chap. 1, Statistical Research Group, Columbia University, New York, McGraw-Hill Book Co., 1947.
[3] N. L. Johnson and B. L. Welch, "Applications of the noncentral t-distribution," *Biometrika*, Vol. 31 (1939–1940), pp. 362–389.
[4] Erich Lehmann, *Theory of Testing Hypotheses*, Mimeographed notes recorded by Colin Blyth, ASUC Bookstore, University of California, Berkeley, 1948–1949.
[5] S. Rushton, "On a sequential t-test," *Biometrika*, Vol. 37 (1950), pp. 326–333.
[6] Abraham Wald, *Sequential Analysis*, John Wiley and Sons, Inc., New York, 1947.

AN EXTENSION OF THE BOREL-CANTELLI LEMMA

By Stanley W. Nash

University of British Columbia

1. Introduction. Consider a probability space $(\Omega, \mathfrak{F}, P)$ and a sequence of events $\{A_n\}$, $A_n \varepsilon \mathfrak{F}$, $n = 1, 2, \cdots$. The upper limiting set of the sequence is defined to be

$$\lim_{n\to\infty} \sup_{k\geqq n} A_k = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k.$$

It is the event that infinitely many of the A_n occur. The purpose of this paper is to find necessary and sufficient conditions for $P(\lim \sup A_n) = 1$.

The general problem of finding the probability of an infinite number of a sequence of events occurring was considered by Borel [1], [2] and Cantelli [3]. In what follows we shall use the following notations. Let $\alpha_n = I(A_n)$, the indicator of the event A_n (or characteristic function of the set A_n), that is

$$\alpha_n = \begin{cases} 1 \text{ when } A_n \text{ occurs} \\ 0 \text{ when } A_n \text{ fails to occur.} \end{cases}$$

Let $P(A_n \mid \alpha_1\alpha_2 \cdots \alpha_{n-1})$ denote the conditional probability of the event A_n, given the outcomes of the previous $n - 1$ trials. When $n = 1$, the expression is taken to represent the unconditional probability $P(A_1)$. The 1912 Borel criterion stated:

> If $0 < p'_n \leqq P(A_n \mid \alpha_1\alpha_2 \cdots \alpha_{n-1}) \leqq p''_n < 1$ for every n, whatever be α_1, $\alpha_2, \cdots, \alpha_{n-1}$, then $\sum_{j=1}^{\infty} p''_j < \infty$ implies that $P(\lim \sup A_n) = 0$, and $\sum_{j=1}^{\infty} p'_j = \infty$ implies that $P(\lim \sup A_n) = 1$.

Cantelli proved that $\sum_{j=1}^{\infty} P(A_j) < \infty$ always implies that $P(\lim \sup A_n) = 0$.

Paul Lévy [4] clarified the general problem by proving the following theorem.

The subset K (or K') of the sample space Ω for which

$$\sum_{j=1}^{\infty} P(A_j|\alpha_1\alpha_2 \cdots \alpha_{j-1}) < \infty \quad (\text{or } = \infty)$$

and the subset H (or H') of Ω for which lim sup A_n fails to occur (or occurs) differ at most by a set of probability 0. In other words $P(KH') = P(K'H) = 0$ and $P(KH) + P(K'H') = 1$. The hypothesis of the theorem proved in the next