GENERALIZATION OF THE THEOREM OF GLIVENKO-CANTELLI

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Let $X_1, X_2, \cdots$ be independent chance variables with the same distribution function $F(x)$. ($F(x)$ is the probability that $X_1 < x$.) The "empiric" distribution function $F_n^*(x)$ of $X_1, \cdots, X_n$ is given by

$$F_n^*(x) = \frac{1}{n} \sum_{i=1}^{n} \psi_x(X_i), \tag{1}$$

where

$$\begin{aligned} \psi_x(a) &= 0, \qquad x \leqq a \\ &= 1, \qquad x > a. \end{aligned}$$

Thus $F_n^*(x)$ is $1/n$ times the number of $X_1, \cdots, X_n$ which are less than x. We define the distance $\delta(G_1, G_2)$ between the two distribution functions G_1 and G_2 as

$$\delta(G_1, G_2) = \sup_x | G_1(x) - G_2(x) |. \tag{2}$$

Let $P\{\ \}$ denote the probability of the relation in braces. The theorem of Glivenko-Cantelli (see, for example [1], page 260) states that

$$P\{\lim_{n \to \infty} \delta(F(x), F_n^*(x)) = 0\} = 1. \tag{3}$$

Let $Y = X_1^1, \cdots, X_1^k, X_2^1, \cdots, X_2^k, \cdots$, ad inf. be a sequence of independent chance variables such that $X_1^i, X_2^i, \cdots$, ad inf. have the same distribution function (say $F_i(x)$), $i = 1, \cdots, k$. Let q_i, $i = 1, \cdots, k$, be real parameters. We shall prove the following generalization of the theorem of Glivenko-Cantelli.

THEOREM. *Let $q = (q_1, \cdots, q_k)$. Let $F(x \mid q)$ be the distribution function of $\sum_{i=1}^k q_i X_1^i$. Let $F_n^*(x \mid q)$ be the empiric distribution function of*

$$\left(\sum_{i=1}^k q_i X_j^i\right), \qquad j = 1, \cdots, n.$$

Then

$$P\{\lim_{n \to \infty} \sup_q \delta(F(x \mid q), F_n^*(x \mid q)) = 0\} = 1. \tag{4}$$

This stronger version of the Glivenko-Cantelli theorem will prove useful in mathematical statistics for the purpose of estimating unknown distribution functions. We have already made use of essentially our result in [2], [3], and [4].

For typographical simplicity we shall carry through the proof for $k = 2$, and leave to the reader the easy verification of the fact that the method is valid for

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