On simplification, this also reduces to $\sum a_{ii} = \sum a_{ij}/n$, the same as obtained from the condition $B = 0$. Thus an important conclusion is reached that whenever the matrix $A = (a_{ij})$ is such that its elements satisfy the relation $\sum a_{ii} = \sum a_{ij}/n$ both the coefficients A and B of the differential equation (4) vanish simultaneously, thus leading to no solution of the problem.

Since cases II and III are excluded by our assumption $\sum a_{ii} \neq \sum a_{ij}/n$, the problem leads uniquely to the solution obtained in (9). Obviously when the matrix $A = (a_{ij})$ is either positive definite or negative definite, the relation $\sum a_{ii} \neq \sum a_{ij}/n$ is always satisfied. Thus the equality $\sum a_{ii} = \sum a_{ij}/n$ may hold only for some indefinite matrices.

COROLLARY. *Let $X_1, X_2, \cdots, X_n$ be identically distributed independent random variables with a finite second moment. If the ratio of the linear functions of random variables given by $(a_1X_1 + \cdots + a_nX_n)/(X_1 + \cdots + X_n)$ is distributed independently of the sum $X_1 + X_2 + \cdots + X_n$ then each X will follow a gamma distribution.*

PROOF. From the statement above, it follows that the conditional expectation of $(a_1X_1 + \cdots + a_nX_n)^2/(X_1 + \cdots + X_n)^2$ for the fixed sum $X_1 + \cdots + X_n$ is equal to its unconditional expectation. Here the elements of the matrix A are given by $a_{ij} = a_ia_j$ for $i,j = 1, 2, \cdots, n$ and they always satisfy the Schwartz's inequality $\sum a_i^2 > (\sum a_i)^2/n$, excluding the trivial case $\sum a_i^2 = (\sum a_i)^2/n$ which is possible when and only when all a_i's are equal, thus reducing the ratio of the linear functions to a constant. Hence the relation $\sum a_{ii} \neq \sum a_{ij}/n$ is always satisfied and the proof follows at once.

REFERENCE

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MATCHING IN PAIRED COMPARISONS

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1. One of the simplest designs for testing the effect of a treatment is the method of paired comparisons: $2n$ subjects are divided into n pairs, and within each pair the treatment is assigned at random to one of the two subjects while the other is used as a control. This method has the reputation of being most effective if the subjects within each pair are as closely matched as possible. We shall show below that while this is true in the situations occurring most commonly in practice, it is not correct universally.

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