AN EXTENSION OF WALD'S THEORY OF STATISTICAL DECISION FUNCTIONS[1]

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1. Introduction. The material of the present paper was developed, during the spring of 1953, primarily to meet pedagogical needs. It is similar to the contents of Chapters 2 and 3 of Wald's book [1]. The results are an extension of Wald's theory in the sense that some requirements of boundedness or even finiteness of the loss function are removed. Moreover, Wald's requirements of equicontinuity are replaced by a requirement of lower semicontinuity of the loss function.

In the first part of the paper it is shown that, under suitable assumptions, the set $\mathfrak{D}$ of all decision functions can be identified with a convex subset of a certain topological vector space. If further assumptions are made on the loss function, the risk functions become lower semicontinuous linear functions defined on $\mathfrak{D}$. It is then easy to give conditions under which $\mathfrak{D}$, or some subset D of $\mathfrak{D}$, is compact.

The next section is devoted to proofs that convexity and compactness of the space of decision functions, together with lower semicontinuity of the risk functions, imply completeness of the intersection of the class of Bayes' solutions in the wide sense with the closure of the class of Bayes' solutions.

The methods of proof differ very little from the methods used by Wald [1], though it has been necessary to use slightly more general topological methods, for instance, to prove compactness instead of sequential compactness. Although it might be possible to extend the proofs given by Wald [1] or Karlin [2], [3] to the case considered here, it is on the whole simpler and shorter to start from the basic elementary lemmas.

2. Assumptions on the decision problem. In this section weakened forms of Assumptions 3.1 to 3.6 of [1] are stated. Let $X = \{X_i\}$ for $i = 1, 2, \cdots$ be a set of random elements, not necessarily real or even vector valued. Let $\mathfrak{X}$ be the space of values of X and let Ω be an arbitrary set of indices. We will suppose that there is given on $\mathfrak{X}$ a σ-field $\mathfrak{A}$ with respect to which all the X_i's are measurable, and that to each $\omega \in \Omega$ corresponds a probability distribution on $\mathfrak{A}$.

If the variables $X_{i_1}, X_{i_2}, \cdots, X_{i_k}$ are observed in this order, we will say that $\lambda = \{i_1, i_2, \cdots, i_k\}$ is observed and restrict the notation λ to ordered sets of indices which can be observed in the order given by λ, the first variable observed being X_{i_1}, the second X_{i_2}, and so on. The variables $\{X_{i_1}, \cdots, X_{i_k}\}$ determine on $\mathfrak{X}$ a smallest σ-field $\mathfrak{A}_\lambda \subset \mathfrak{A}$ with respect to which they are measurable. For all practical purposes it is equivalent to say that an $\mathfrak{A}$-measurable function $f(x)$ is $\mathfrak{A}_\lambda$-measurable or that it is a function of $\{X_{i_1}, \cdots, X_{i_k}\}$ only.

Received May 25, 1953, revised May 3, 1954.

[1] This paper was prepared with the partial support of the Office of Naval Research.