ON THE FOURIER SERIES EXPANSION OF RANDOM FUNCTIONS[1]

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Problems involving stationary stochastic processes are often treated by approximating the original processes by Fourier series with orthogonal random coefficients.[2] In this paper we justify this technique in certain instances.

We let $x(t)$ denote a real- or complex-valued stochastic process defined for all values of t. We assume the first and second moments of $x(t)$ exist. We write $R(s, t) = E\{x(s)\overline{x(t)}\}$ and in case $R(s, t)$ is a function of $(t - s)$ only (that is, $x(t)$ is stationary in the wide sense), we write $\rho(t - s) = R(s, t)$. We assume everywhere that $E\{x(t)\} = 0$.

We define the stochastic process $x(t)$ to be *periodic* if the random variables $x(t_1)$ and $x(t_1 + T)$ are equal with probability one for all t_1 and some constant T. If $x(t)$ is periodic, then $R(s, t)$ is periodic in each variable. If $x(t)$ is wide-sense stationary, then it is periodic if and only if $\rho(\tau)$ is periodic.

Our first result follows from the theorem due independently to Karhunen and Loève which states: Let $x(t)$ be continuous in the finite interval (a, b), then

$$x(t) = \underset{n\to\infty}{\text{l.i.m.}} \sum_1^n x_i \psi_i(t)$$

where the $\psi_i(t)$ form an orthonormal system over (a, b) and where $E\{x_i\overline{x_j}\} = \lambda_i\delta_{ij}$ if and only if the $\psi_i(t)$ and the λ_i are a system of eigenfunctions and eigenvalues of the integral equation

$$\int_a^b R(s, t)\overline{\psi(t)}\, dt = \lambda\overline{\psi(s)}.$$

THEOREM 1. *Let $x(t)$ be a wide-sense stationary stochastic process continuous in mean square. Then*

$$x(t) = \text{l.i.m.} \sum_{k=-n}^{n} \frac{x_i e^{ik\omega\tau}}{\sqrt{T}}, \qquad \omega = \frac{2\pi}{T},$$

on the interval $(0, T)$, *where the x_i are pairwise orthogonal if and only if $x(t)$ is periodic with period T.*

PROOF. From the Karhunen-Loève theorem and the remark above about periodicity it follows that we need to show that

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[2] Of course, the use of Fourier series can usually be avoided by the use of the spectral representation theorem for stationary processes. See [1] p. 527.