CALCULATION OF EXACT SAMPLING DISTRIBUTION OF RANGES FROM A DISCRETE POPULATION[1]

By Irving W. Burr

Purdue University

1. Introduction. The exact sampling distribution for ranges is known for but few populations, and general information on moments of the range is incomplete. This note gives a method for calculating the exact sampling distribution for discrete universes having a finite range and approximating those for populations with an infinite range.

2. Derivation. Consider a random variable X defined on integers a to b, both finite. Let p_i be the probability that X is i, and $p(R)$ be the probability that the range takes the value R. Then for a sample of n X's from the population (drawn with replacement) we have

$$p(R) = \sum_{i=a}^{b-R} \sum_{r=1}^{n-1} \sum_{s=1}^{n-r} \frac{n!\, p_i^r p_{i+R}^s}{r!\, s!\, (n-r-s)!} (p_{i+1} + \cdots + p_{i+R-1})^{n-r-s}, \tag{1}$$

since the summand contains at least one X at i and at least one X at $i + R$ and those X's not at these values are all between, and the summation is over all possible such samples. To obtain a more useful form we let

$$M(i, R) = \sum_{j=1}^{i+R} p_j. \tag{2}$$

Then

$$\begin{aligned} p(R) &= \sum_{i=a}^{b-R} \sum_{r=1}^{n-1} \sum_{s=1}^{n-r} \frac{n!\, p_i^r p_{i+R}^s}{r!\, s!\, (n-r-s)!} M^{n-r-s}(i+1, R-2) \\ &= \sum_{i=a}^{b-R} [\text{terms of } M^n(i, R) \text{ containing at least one } i \text{ and at least one } i + R]. \end{aligned}$$

To get the desired terms of $M^n(i, R)$, we first subtract from it all of those terms which fail to contain any $i + R$, namely, $M^n(i, R - 1)$. Then we also subtract off those which fail to contain any i, namely $M^n(i + 1, R - 1)$. But these two expressions overlap to the extent of $M^n(i + 1, R - 2)$, that is, terms with neither i nor $i + R$. So this must be added back on. Thus we have

$$\begin{aligned} p(R) = \sum_{i=a}^{b-R} [M^n(i, R) - M^n(i, R-1) \\ - M^n(i+1, R-1) + M^n(i+1, R-2)]. \end{aligned} \tag{3}$$

Received November 27, 1953; revised July 29, 1954.

[1] Presented by title to the Institute, December 27, 1951, at Boston.