ON THE STOCHASTIC INDEPENDENCE OF TWO SECOND-DEGREE POLYNOMIAL STATISTICS IN NORMALLY DISTRIBUTED VARIATES

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A remarkable property of the normal law as proved by Craig [1] is that if $x_1, x_2, \cdots, x_n$ are n identically and independently distributed normal variates each with zero mean and unit variance, then the necessary and sufficient condition for the stochastic independence of two real homogeneous quadratic statistics $Q_1 = xAx'$ and $Q_2 = xBx'$ is that the matrix product $AB = 0$. The same theorem has also been proved independently by Hotelling [2], Sakamoto [5], Matusita [3], and Ogawa [4].

In the present paper we shall establish a corresponding theorem for the case of two second-degree polynomial statistics in normally distributed variates, and give some related results.

Theorem 1. *Let $x_1, x_2, \cdots, x_n$ be n independently and identically distributed normal variates each with zero mean and unit variance; then the necessary and sufficient condition that two real polynomial statistics of the second degree denoted by $P_1 = xAx' + lx'$ and $P_2 = xBx' + mx'$ are stochastically independent is that*

$$\text{(i) } AB = 0, \qquad \text{(ii) } lB = 0, \qquad \text{(iii) } mA = 0, \qquad \text{(iv) } lm' = 0.$$

Here, x, l, and m, respectively, represent the row-vectors $(x_1, x_2, \cdots, x_n)$, $(l_1, l_2, \cdots, l_n)$, and $(m_1, m_2, \cdots, m_n)$ and x', l', and m', as usual, represent their corresponding transposes and $A = (a_{ij})$ and $B = (b_{ij})$ are both real symmetric matrices of order n.

Proof of Sufficiency. Without any loss of generality we can write t_1 and t_2 in place of it_1 and it_2, respectively, so that the characteristic function of the joint distribution of P_1 and P_2 is given by

$$\phi(t_1, t_2) = E[\exp(t_1P_1 + t_2P_2)]. \tag{1}$$

Hence,

$$\begin{aligned}\phi(\tfrac{1}{2}t_1, \tfrac{1}{2}t_2) &= E[\exp(\tfrac{1}{2}t_1P_1 + \tfrac{1}{2}t_2P_2)] \\ &= |I - t_1A - t_2B|^{1/2} \\ &\qquad \times \exp\{\tfrac{1}{8}(t_1l + t_2m)(I - t_1A - t_2B)^{-1}(t_1l + t_2m)'\}.\end{aligned} \tag{2}$$

Now, putting $t_2 = 0$ and $t_1 = 0$ alternatively in (2), we get

$$\phi(\tfrac{1}{2}t_1, 0) = |I - t_1A|^{1/2} \exp[\tfrac{1}{8}t_1^2 l(I - t_1A)^{-1}l'], \tag{3a}$$

$$\phi(0, \tfrac{1}{2}t_2) = |I - t_2B|^{1/2} \exp[\tfrac{1}{8}t_2^2 m(I - t_2B)^{-1}m'], \tag{3b}$$

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