ASYMPTOTIC DISTRIBUTIONS OF TWO GOODNESS OF FIT CRITERIA

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1. Results. Let $\{X_1, X_2, \dots\}$ be a stochastic process in which each random variable takes as values only the integers $1, 2, \cdots, s$. To test the null hypothesis that the process is independent and stationary with $P\{X_n = k\} = p_k > 0$, it is natural to form the statistic

$$\sum_{u_1, \cdots, u_\nu = 1}^{s} \frac{(n_{u_1 \cdots u_\nu} - np_{u_1} \cdots p_{u_\nu})^2}{np_{u_1} \cdots p_{u_\nu}}, \tag{1.1}$$

where $n_{u_1 \cdots u_\nu}$ is the number of integers $m \leqq n$ for which $(X_m, \cdots, X_{m+\nu-1})$ is the ν-tuple $(u_1, \cdots, u_\nu)$. In Section 2 we show that under the null hypothesis the distribution function of (1.1) approaches, as $n \to \infty$, the distribution function

$$\mathop{*}_{\lambda=1}^{\nu-1} K_{s^{\nu-1-\lambda}(s-1)^2}(x/\lambda) * K_{s-1}(x/\nu), \tag{1.2}$$

where $K_i(x)$ is the chi-square distribution with i degrees of freedom and the first $*$ denotes iterated convolution in the obvious way. Good [1], using different methods, has obtained this result for the special case in which the p_k are all equal and s is a prime number.

If the p_k are estimated by n_k/n, there results the statistic

$$\sum_{u_1, \cdots, u_\nu = 1}^{s} \frac{(n_{u_1 \cdots u_\nu} - n^{1-\nu} n_{u_1} \cdots n_{u_\nu})^2}{n^{1-\nu} n_{u_1} \cdots n_{u_\nu}}. \tag{1.3}$$

In Section 3 we show that under the hypothesis that $\{X_n\}$ is stationary and independent, the distribution function of (1.3) approaches, as $n \to \infty$, the distribution function

$$\mathop{*}_{\lambda=1}^{\nu-1} K_{s^{\nu-1-\lambda}(s-1)^2}(x/\lambda). \tag{1.4}$$

In the special case $\nu = 2$ this result is implicit in the work of Hoel [2]. Note that in this case (1.4) becomes $K_{(s-1)^2}(x)$.

The means and variances of the distributions (1.2) and (1.4) are easily written down. It is obvious that if ν is fixed and $s \to \infty$, then these distributions are, when normed by their means and standard deviations, asymptotically normal. It is a simple matter to show, using Ljapunov's condition and the fact that the distributions are convolutions, that the same thing is true if s is fixed and $\nu \to \infty$. By interpolation in the tables of [3] one can get an approximation to (1.2) for the case $s = 2$ and $\nu = 2$ and an approximation to (1.4) for the case

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