ON THE SERIAL TEST FOR RANDOM SEQUENCES

By I. J. Good

Let $a_1, a_2, \cdots, a_N$ be a finite random sequence, $\mathfrak{S}$, of independent random variables with $P(a_j = r) = p_r (j = 1, 2, \cdots, N; r = 0, 1, \cdots, t - 1)$, where $\sum p_r = 1$. We call each of $0, 1, \cdots, t - 1$ "digits." Associated with $\mathfrak{S}$ is the corresponding cyclic sequence $\bar{\mathfrak{S}}$, defined by regarding the first digit of $\mathfrak{S}$ as immediately following the last one. We shall always denote properties of $\bar{\mathfrak{S}}$ by placing a bar over the corresponding algebraic symbol relating to $\mathfrak{S}$.

A sequence of ν digits is called a ν-sequence. A ν-sequence is said to belong to $\mathfrak{S}$ if it is of the form $a_j, a_{j+1}, \cdots, a_{j+\nu-1} (j = 1, 2, \cdots, N - \nu + 1)$ and to belong to $\bar{\mathfrak{S}}$ if j is also allowed to take the values $N - \nu + 2, N - \nu + 3, \cdots N$, where a_{N+k} is identified with a_k for all integers k. Let $n_{r_1, r_2, \cdots, r_\nu}$, or $n_{\mathfrak{r}}$ for short, be the number of ν-sequences in $\mathfrak{S}$ which are the ν-sequence $(r_1, r_2, \cdots, r_\nu) = \mathfrak{r}$ (where $r_1, r_2, \cdots, r_{\nu-1}$ and r_ν are digits). Let

$$p_{\mathfrak{r}} = p_{r_1} \cdots p_{r_\nu}, \tag{1}$$

$$\psi_\nu^2 = \sum_{\mathfrak{r}} \frac{[n_{\mathfrak{r}} - (N - \nu + 1)p_{\mathfrak{r}}]^2}{(N - \nu + 1)p_{\mathfrak{r}}}, \tag{2}$$

$$\bar{\psi}_\nu^2 = \sum_{\mathfrak{r}} \frac{(\bar{n}_{\mathfrak{r}} - Np_{\mathfrak{r}})^2}{Np_{\mathfrak{r}}}, \tag{3}$$

where $\mathfrak{r}$ runs through all its t^ν possible values. Let

$$\psi_0^2 = \bar{\psi}_0^2 = 1. \tag{4}$$

We shall prove that if[1]

$$\nu \leqq \tfrac{1}{2}(N + 1), \tag{5}$$

then

$$\mathcal{E}(\psi_\nu^2) = t^\nu - 1, \tag{6}$$

and

$$\mathcal{E}(\bar{\psi}_\nu^2) = t^\nu - 1. \tag{7}$$

The special case $p_0 = p_1 = \cdots = p_{t-1} = t^{-1}$ was dealt with by Good [1], who also proved some asymptotic results that have been generalized to arbitrary random sequences by Billingsley [2]. When we apply these asymptotic results to actual (finite) values of N, our confidence in their approximate validity is increased on finding that equations (6) and (7) are exact.

We mention in passing that the variances of ψ_ν^2 and $\bar{\psi}_\nu^2$ are asymptotically

$$\frac{2}{t - 1}(t^{\nu+1} + t^\nu - (2\nu + 1)t + 2\nu - 1), \tag{8}$$

but we do not know how good the approximation is for actual values of N.

Received April 4, 1956.

[1] The condition (5) was also required by Good [1], but in error it was not mentioned.