NOTES

WAITING TIMES WHEN QUEUES ARE IN TANDEM

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1. We study the distribution of waiting times when customers proceed to a second (multiple-counter) queue after having been processed at a first (multiple-counter) queue[1]. For reasons of expediency we restrict ourselves to the case of unsaturated queues in "equilibrium," that is, to stationary statistics. The main results are for the case of exponential service time, where it turns out that, contrary to a-priori intuition, the situation is surprisingly simple. As shown by Theorem 6, no such simple behavior can be expected when the service time distributions are even only slightly more general. Theorem 4 was first found essentially by P. J. Burke [1], by a different method.[2]

The concept of reversibility of a Markov chain, certain aspects of which are discussed in Sec. 2, has turned out to be fruitful in connection with the analysis, and is of some independent interest.

2. A stationary stochastic process $N(t)$ is said to be reversible if $N(t)$ and $N(-t)$ have the same multivariate distributions. If $N(t)$ is a discrete or continuous parameter Markov chain with a denumerable state space, say, $0, 1, 2, \cdots$, then $N(-t)$ is a process of the same type. The necessary and sufficient condition for reversibility becomes

$$(1) \qquad \theta_{ij}(t) = p_i P_{ij}(t) = p_j P_{ji}(t) = \theta_{ji}(t), \qquad i, j = 0, 1, 2, \cdots,$$

where p_i and $P_{ij}(t)$ are respectively, the stationary, and transition probabilities of $N(t)$.

Kolmogorov's criterion for reversibility of Markov chains with a finite state space ([8]; [5], p. 66) may, in a special case, be immediately generalized to the denumerable state-space case, as follows.

Theorem 1. *Let* $N(k)$, $k = 0, \pm 1, \pm 2, \cdots$, *be an irreducible stationary discrete-parameter Markov chain with the state space* $0, 1, 2, \cdots$, *the stationary probabilities* u_k, *and the singlestep transition probabitilies* π_{ij}. *A necessary and sufficient condition for the reversibility of* $N(k)$ *is that*

$$(2) \qquad \pi_{i_1 i_2}\pi_{i_2 i_3} \cdots \pi_{i_{n-1} i_n}\pi_{i_n i_1} = \pi_{i_1 i_n}\pi_{i_n i_{n-1}} \cdots \pi_{i_3 i_2}\pi_{i_2 i_1}$$

for every sequence of non-negative integers $(i_1, i_2, \cdots, i_n, i_1)$ *beginning and ending with the same integer.*

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[1] A part of this paper represents work done at The RAND Corporation.

[2] A special case of a part of this theorem was also treated (unpublished) by H. H. Goode and R. E. Machol. Their work is to appear in a text.