THE DISTRIBUTION OF THE RATIOS OF CERTAIN QUADRATIC FORMS IN TIME SERIES[1]

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1. Introduction. In testing the hypothesis that successive members of a series of observations are serially correlated a number of statistics have been proposed. Durbin and Watson [4] gave the exact distribution of several of these statistics when they are slightly modified. We shall extend the work of Durbin and Watson for a non-null case of two of their modified statistics and also find a simple expression for the moments of another of their statistics.

2. The Double root result. Assume that $X' = (x_1, x_2, \cdots, x_n)$ has probability density

$$f(X) = |\Lambda|^{1/2}(2\pi)^{-n/2} \exp[-X'\Lambda X/2], \tag{2.1}$$

where Λ is a positive definite matrix and $n = 2m$. Let

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_1 \end{pmatrix}; \quad B = \begin{pmatrix} B_1 & 0 \\ 0 & B_1 \end{pmatrix}; \quad \Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_1 \end{pmatrix}; \tag{2.2}$$

where B_1 is positive definite or positive semi-definite and of rank $m - q$ which is $\geqq$ the rank of A_1, a real symmetric matrix. Further assume that A, B, and Λ commute pairwise, and that the characteristic roots a_j of A and the characteristic roots b_j of B are so numbered that if $a_j \neq 0, b_j > 0$ and $a_j/b_j \geqq a_{j+1}/b_{j+1}$ for all a_j and a_{j+1} which are $\neq 0$.

Now

$$G(z) = P\left[\frac{X'AX}{X'BX} \leqq z\right] = P[X'(A - zB)X \leqq 0], \tag{2.3}$$

where X is $N(0, \Lambda^{-1})$. Making an orthogonal transformation $X = PY$ where $P'AP = D_a$, $P'BP = D_b$, $P'\Lambda P = D_\lambda$ are diagonal matrices with elements $a_j = a_{m+j}$, $b_j = b_{m+j}$ and $\lambda_j = \lambda_{m+j}$, we get

$$G(z) = P[Y'(D_a - zD_b)Y \leqq 0], \tag{2.4}$$

where Y is $N(0, D_\lambda^{-1})$. Now let $Y = D_\lambda^{-1/2}W$ so that

$$G(z) = P[W'(D_a - zD_b)D_\lambda^{-1}W \leqq 0], \tag{2.5}$$

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