IDEMPOTENT MATRICES AND QUADRATIC FORMS IN THE GENERAL LINEAR HYPOTHESIS

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1. Introduction. The important role that idempotent matrices play in the general linear hypothesis theory has long been recognized ([1], [2]), but their usefulness seems not to have been fully exploited. The purpose of this paper is to state and prove some theorems about idempotent matrices and to point out how they might be used to advantage in linear hypothesis theory.

2. Notation and Definitions. Throughout this paper an idempotent matrix will mean a symmetric matrix A such that $AA = A$ (for the sake of brevity we will use the word idempotent matrix to indicate a symmetric idempotent matrix unless specifically stated otherwise). The theorems will not necessarily hold for nonsymmetric idempotent matrices. The statement: Y is distributed as $N_p(\mu, V)$, will mean that a $(p \times 1)$ random vector Y has the p-variate normal distribution whose mean is the $(p \times 1)$ vector, μ, and whose covariance matrix is the positive definite symmetric matrix, V. The statement: u is distributed as $\chi^2(n)$ will mean that a scalar random variable u has the Chi-square distribution with n degrees of freedom, and the statement: v is distributed as $\chi'^2(n, \lambda)$ will mean that the scalar random variable v is distributed as the noncentral Chi-square distribution with n degrees of freedom and with noncentrality, λ. The frequency function of v is ([3])

$$f(v) = \sum_{i=0}^{\infty} \frac{e^{-\lambda}\lambda^i}{i!} \cdot \frac{v^{n+2i-2/2}e^{-(v/2)}}{2^{n+2i/2}\Gamma\left(\frac{n+2i}{2}\right)}, \qquad 0 \leqq v < \infty.$$

If $\lambda = 0$, then the noncentral Chi-square distribution degenerates into the central Chi-square distribution.

A' will indicate the transpose of the matrix A, and A^{-1} will indicate the inverse. I_p will indicate the $(p \times p)$ identity matrix and φ will indicate a null matrix. Below is a list of well-known theorems which will be needed in the succeeding sections.

THEOREM A. *If A is an $(n \times n)$ symmetric matrix of rank p, then a necessary and sufficient condition that A is idempotent is that each of p of the characteristic roots of A is equal to unity and the remaining $(n - p)$ characteristic roots are equal to zero.*

THEOREM B. *If A is an idempotent matrix, then the rank of A equals the trace of A.*

THEOREM C. *The only nonsingular idempotent matrix is the identity matrix.*

THEOREM D. *If A is an $(n \times n)$ idempotent matrix of rank p such that $p < n$ $(p = n)$, then A is a positive semidefinite matrix (positive definite matrix).*

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