CONTRIBUTIONS TO THE THEORY OF RANK ORDER STATISTICS—THE "TREND" CASE[1]

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0. Introduction. In spirit this paper is a continuation of [5] and the techniques and terminology developed there will be used. Here we are concerned with the detailed relationships between the probabilities of rank orders under various "trend" hypotheses. The relationships found are of interest in themselves and in the theory of nonparametric tests of hypotheses.

Typically we shall be concerned with mutually independent random variables $X_1, \cdots, X_N$ such that X_i has a distribution function of the form $F(x - \theta_i)$ where the θ_i form an increasing sequence. Conditions are given under which one rank order is always more probable than another, one rank order is equally probable with another, and these results are translated into conditions for admissible rank order tests. References [1], [6], and [7] summarize information regarding large sample properties of nonparametric tests of this type of hypothesis.

In Section 1 two definitions of rank order are presented along with some "algebraic" properties of rank orders. Section 2 contains an enumeration of the hypotheses that we are concerned with. Section 3 presents theory and Section 4 contains applications.

1. Rank orders.

DEFINITION: The rank order corresponding to the N distinct numbers $x_1, \cdots, x_N$ is the vector $r = (r_1, \cdots, r_N)$ where r_i is the number of x_j's $\leqq x_i$. r is a permutation of the first N integers. If in the definition the x's are replaced by random variables then R will be used instead of r. R will be defined with probability one when the underlying random variables have continuous distributions.

DEFINITION: $r' L_{ij} r$ if $r'_k = r_k$ for $i \neq k \neq j$; $r'_i = r_j$, $r'_j = r_i$; and

$$(r_i - r_j)(i - j) > 0.$$

Thus, if $r = (2, 3, 6, 5, 4, 1)$ and $r' = (2, 5, 6, 3, 4, 1)$ then $r' L_{24} r$. We shall write $r' L r$ as an abbreviation for $r' L_{ij} r$ or to denote that there is a chain of rank orders $r^1, \cdots, r^t, \cdots, r^T$ such that $r' L_{i_0 j_0} r^1 L_{i_1 j_1} \cdots r^t L_{i_t j_t} r^{t+1} \cdots r^T L_{i_T j_T} r$. Thus if $r = (2, 3, 6, 5, 4, 1)$ and $r' = (3, 5, 6, 4, 2, 1)$ then $r' L r$, $T = 2$ and $r' L_{15}(2, 5, 6, 4, 3, 1) L_{45}(2, 5, 6, 3, 4, 1) L_{24} r$. For many of the hypotheses (see

Received November 8, 1956; revised July 18, 1957.

[1] This work was supported by an Office of Naval Research contract at Stanford University.

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