THE KOLMOGOROV-SMIRNOV, CRAMÉR-VON MISES TESTS

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1. Preface. This is an expository paper giving an account of the “goodness of fit” test and the “two sample” test based on the empirical distribution function—
tests which were initiated by the four authors cited in the title. An attempt is
made here to give a fairly complete coverage of the history, development, present
status, and outstanding current problems related to these topics.

The reader is advised that the relative amount of space and emphasis allotted
to the various phases of the subject does not reflect necessarily their intrinsic
merit and importance, but rather the author’s personal interest and familiarity.
Also, for the sake of uniformity the notation of many of the writers quoted has
been altered so that when referring to the original papers it will be necessary to
test check their nomenclature.

2. The empirical distribution function and the tests. Let $X_1, X_2, \ldots, X_n$ be
independent random variables (observations) each having the same distribution
function $U(x) = \Pr\{X_i < x\}$ and put

\[
\epsilon(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0. \end{cases}
\]

Then the (random) function

\[
F_n(x) = \frac{1}{n} \sum_{j=1}^n \epsilon(x - X_j)
\]

is called the empirical distribution function of the data. Clearly $F_n(x)$ is the pro-
portion of the $X_i, i = 1, 2, \ldots, n$, which are less than $x$.

It is easy to calculate the first and second order moments

\[
E(F_n(x)) = U(x),
\]

\[
\text{Cov}(F_n(x), F_n(y)) = E(F_n(x)F_n(y)) - U(x)U(y) = \frac{1}{n} c(U(x), U(y)),
\]

where

\[
c(s, t) = \min(s, t) - st = \begin{cases} s(1 - t) & s \leq t \\ t(1 - s) & s \geq t, \end{cases}
\]

We quote a few classical consequences of the definition (2.2):

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