NOTES

AN EXTENSION OF THE OPTIMUM PROPERTY OF THE SEQUENTIAL PROBABILITY RATIO TEST

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Let $f(x, \theta)$ be a family of densities or discrete probability functions depending on the parameter θ. Let H_0 be the hypothesis $\theta = \theta_0$ and H_1 the hypothesis that $\theta = \theta_1$. A sequential probability ratio test of H_0 versus H_1 is defined by two numbers A and B. After drawing the mth observation, sampling is continued if

$$B < \prod_{i=1}^{m} \frac{f(x_i, \theta_1)}{f(x_i, \theta_0)} < A, \tag{1}$$

where $x_1, \cdots, x_m$ are the first m observations. If the probability ratio is at least equal to A, H_1 is accepted, and if it is not greater than B, H_0 is accepted.

For any sequential procedure T, let the operating characteristic be

$$L(\theta, T) = \Pr \{\text{Accepting } H_0 \mid \theta, T\}, \tag{2}$$

and let $\mathcal{E}_\theta(n \mid T)$ be the expected number of observations required by T when sampling from $f(x, \theta)$. The so-called optimum property (see [5], for instance) of a sequential probability ratio test, say T^*, is that if $L(\theta_0, T) \geqq L(\theta_0, T^*)$ and $L(\theta_1, T) \leqq L(\theta_1, T^*)$, then

$$\mathcal{E}_{\theta_0}(n \mid T) \geqq \mathcal{E}_{\theta_0}(n \mid T^*), \qquad \mathcal{E}_{\theta_1}(n \mid T) \geqq \mathcal{E}_{\theta_1}(n \mid T^*).$$

In many cases this optimum property can be extended to all values of the parameter. Suppose $\theta_0 < \theta_1$, and let $\bar{\theta}$ be a number to be defined later such that $\theta_0 < \bar{\theta} < \theta_1$. Under conditions stated below, we give the extended optimum property. If

$$\begin{aligned} L(\theta, T) &\geqq L(\theta, T^*), \qquad \theta < \bar{\theta}, \\ L(\theta, T) &\leqq L(\theta, T^*), \qquad \theta > \bar{\theta}, \end{aligned} \tag{3}$$

for all $\theta \neq \bar{\theta}$, then

$$\mathcal{E}_\theta(n \mid T) \geqq \mathcal{E}_\theta(n \mid T^*) \tag{4}$$

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[1] The result reported in this note was mentioned by the late M. A. Girshick to several of his colleagues, but was unpublished at the time of his death. Since I think the result is of sufficient interest to be in the literature, I have taken the liberty of writing this note in Girshick's name. T. W. Anderson.

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