TRANSIENT ATOMIC MARKOV CHAINS WITH A DENUMERABLE NUMBER OF STATES[1]

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1. Introduction. Many of the more interesting transient Markov chains have the property that for any set of states A and any initial distribution, the probability of entering A infinitely often (i.o.) is either zero always or one always. This type of chain has been termed *atomic* by D. Blackwell [1] and is exemplified by the three-dimensional random walk or by the successive sums of independent, identically distributed random variables.

In this paper we investigate the "fine structure" of an atomic chain, that is, we try to characterize the class of all sets A such that $P(x_n \varepsilon A \text{ i.o.}) = 0$. The study is restricted to atomic chains with a countable set of states which, for convenience of notation, we identify with the integers, and with stationary transition probabilities $p_{ij}^{(n)}$.

The martingale convergence theorem is used in [1] to show that a necessary and sufficient condition for atomicity is that every bounded solution ϕ of

$$\phi(i) = \sum_j p_{ij}\phi(j)$$

be constant. We use as our main tool the semi-martingale convergence theorem and the corresponding equation $\phi(i) \geqq \sum_j p_{ij}\phi(j)$ and obtain a complete, but not simple, characterization of the fine structure of transient atomic chains.

To illustrate the use of the above characterization we prove two theorems regarding the return to equilibrium times $x_0, x_1, \cdots$ in the coin-tossing game. The latter of these is then used to prove that there exists no set of numbers $\{\lambda_m\}$ such that[2] $P(x_n \varepsilon A \text{ i.o.}) = 0 \Leftrightarrow \sum_{m \varepsilon A} \lambda_m < \infty$.

This last result shows that, in general, there is no simple resolution to the question of defining the fine structure. There are, however, a number of interesting transient atomic chains which have the property that every infinite set of states is entered infinitely often with probability one. These chains are the subject of papers by Chung and Derman [2], and Breiman [3].

2. Use of the semi-martingale theorem.

Theorem 1. *Let $x_0, x_1, \cdots$ be an atomic chain. Then for ϕ any nonnegative solution of*

(a) $$\phi(i) \geqq \sum_j p_{ij}\phi(j)$$

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[2] The referee has informed us that a similar theorem for the three-dimensional random walk has been proved by P. Erdös and B. J. Murdoch (unpublished).