SEMIMARTINGALES OF MARKOV CHAINS

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1. Introduction. We shall deal throughout this paper with absorbing Markov chains with a finite number of states. An absorbing Markov chain is one that has a set of "boundary" states which once reached cannot be left, and such that from any state the process reaches the boundary with probability 1. The chain is given by the transition matrix P, with entries p_{ij}.

More precisely, a state i is a "boundary" state if $p_{ii} = 1$. The remaining states will be called "interior" states. We must require that it is possible to reach the boundary from every interior state, not necessarily in one step. We assume, that there are r absorbing states and s interior states. The set of boundary states will be called B, the set of interior states I.

An upper semimartingale is a function on the states of the chain, such that the expected value of the function after one step from any state is greater than or equal to the value of the function at the state. A lower semimartingale is defined similarly, with the inequalities reversed. A martingale is a function on the states that is both an upper and a lower semimartingale.

A function on the states can be conveniently represented by a column vector. Such a vector z is an *upper semimartingale* if $Pz \geqq z$, a *lower semimartingale* if $Pz \leqq z$, and a *martingale* if $Pz = z$.

We assume that a set of nonnegative *boundary values* is assigned to the elements of B, v_j being assigned to state j. We denote by U the set of all nonnegative upper semimartingales and by U^* the set of all nonnegative lower semimartingales having the right boundary values. Thus U is the set of all vectors such that

$$\text{(a)}\ Pz \geqq z, \quad \text{(b)}\ z \geqq 0, \quad \text{(c)}\ \{z\}_j = v_j \quad \text{for} \quad j \varepsilon B.$$

The set U^* consists of the vectors satisfying conditions (b) and (c), and condition (a) with the inequality sign reversed.

Throughout the paper $\{z\}_j$ will denote the jth componnet of the vector z. Inequality signs between vectors will assert that the inequality holds componentwise.

A representation theorem will be developed for all nonnegative semimartingales with the prescribed boundary values in terms of martingales of modified chains. A modified chain is one obtained by adding interior states to the boundary, and assigning value 0 to them. The representation is unique and leads to a simple geometric interpretation. U will be represented (except in certain de-

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