LINEAR ESTIMATION FROM CENSORED DATA

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1. Introduction. Suppose that a sample of n random variables is taken from a continuous probability distribution, whose density function is $f\,[(y - \mu)/\sigma]/\sigma$, where μ and σ are unknown. Arrange the variables in order of magnitude, and denote them by $y_1, y_2, \cdots, y_n$, where

$$y_1 < y_2 < \cdots < y_n .$$

We shall discuss the problem of estimating μ and σ from the k successive variables $y_u, y_{u+1}, \cdots, y_v$, where $v = u + k - 1$. This problem arises, for example, in life-testing, and some applications are described by Gupta [7].

When using the principal results derived here, the expected values of ordered variables are essential, but tables of these quantities for normal samples are, at present somewhat limited. However, recent studies by Berkson [1] have shown the importance of the logistic distribution, which closely resembles the normal, and some properties of ordered logistic variables are given in Section 2. We now turn to the main problem. If u and v are fixed, the best linear unbiased estimates of μ and σ can be calculated by least squares, given the expected value and dispersion matrix of the vector of ordered variables (Godwin [6], Lloyd [11], Gupta [7]. In general, special tables become necessary, and it seems desirable to obtain simple formulae when samples are moderate or large in size. This is achieved in Section 3, where asymptotic values of the coefficients of $y_u, y_{u+1}, \cdots, y_v$ are derived. An examination of the conditions involved is supplied in Section 4, by considering the limiting form of the maximum likelihood equations. Several illustrative numerical tables complete the paper.

2. Ordered logistic variables. The logistic distribution is defined by

$$L = \log\{p/(1 - p)\}, \tag{1}$$

where p is the probability of a value less than L. Suppose that $L(i; n)$ is the ith variable in ascending order in a sample of size n from this distribution. Then

$$\begin{aligned} \mathcal{E} \exp\{wL(i; n)\} &= \frac{n!}{(i-1)!\,(n-i)!} \int_0^1 \left(\frac{p}{1-p}\right)^w p^{i-1}(1-p)^{n-i}\,dp \\ &= \frac{(i-1+w)!\,(n-i-w)!}{(i-1)!\,(n-i)!}. \end{aligned} \tag{2}$$

Take logarithms, differentiate with respect to w, and put $w = 0$. The cumulants of $L(i; n)$ are

$$\kappa_j(i; n) = \frac{d^j}{dw^j} \log (i - 1)! + (-1)^j \frac{d^j}{dw^j} \log (n - i)! \tag{3}$$

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