A SMOOTH INVERTIBILITY THEOREM[1]

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1. Introduction. In connection with discussions of fiducial inference (e.g., see 3), it is often desirable to consider the invertibility of certain mappings. We shall say that a mapping is smoothly invertible (of class α) if (condition (1) of [3] is irrelevant here):

(2) the mapping is 1-1 and hence has a single-valued inverse,

(3) this inverse is a continuous function (and has continuous derivatives of all orders up to α).

All too often, as has been emphasized to the author by L. J. Savage, the question of invertibility has been "answered" by showing that a Jacobian is of constant sign. It is, of course, well known that this does not suffice to give uniqueness in the large.

Explicit conditions sufficient for uniqueness in the large do not seem to be given frequently in the literature. The present note records an explicit theorem in a form which seems likely to be of service in such conditions.

2. A smooth invertibility theorem. We now state the smooth invertibility theorem as follows:

Any α times continuously differentiable mapping from an arcwise connected open domain (in n dimensions) *to a simply connected range, whose Jacobian determinant is continuous and of one sign throughout the domain, and whose inverse carries compact sets into compact sets is smoothly invertible of order α.*

In our application it is convenient to use the

Observation. *If the open domain and the simply connected range are both the whole plane (or the whole of any Euclidian space) then the inverse will carry compact sets into compact sets provided that it carries bounded sets into bounded sets.*

The proof of this observation follows immediately from the remarks that (i) the inverse image of closed sets by a continuous mapping are always closed, (ii) in the whole plane the compact sets are just those which are closed and bounded.

3. Proof. The proof of the theorem rests on a classical result about local inversion (which makes no use of arcwise connectedness, simple connectedness, or the hypothesis about the inverse taking compact sets into compact sets) and a purely topological result relating local uniqueness of inverses to global uniqueness (which makes no explicit use of the differentiability conditions).

We say that N is a local inverse neighborhood of x if

(1) $f(N)$ is a neighborhood of $y = f(x)$,

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