ON THE INTEGRODIFFERENTIAL EQUATION OF TAKÁCS. I.

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1. Introduction. This paper is devoted to a study of certain aspects of the mixed-type Markov process $\eta(t)$, originally treated by Takács [8]. It extends and unifies a number of results of previous workers.

Let $N(t)$, $N(0) = 0$, $t \geqq 0$ denote max $\{n \mid t_n \leqq t\}$. We shall be especially interested in the case where $0 < t_1 < t_2 < \cdots$ are the events of an (in general) non-homogeneous Poisson process of density $\lambda(t) \geqq 0$. We assume that $\lambda(t)$ is Riemann integrable over all finite intervals. (The homogeneous Poisson process corresponds to $\lambda(t) = \text{const.}$) Let $\chi_0, \chi_1, \chi_2, \cdots$ be a sequence of non-negative random variables. Except in a part of Section 5, they are mutually independent, and independent of $N(t)$; moreover, $H(x) = \Pr\{\chi_i \leqq x\}$ is the same for $i = 1, 2, \cdots$. Introducing the notations

$$\int_{-\infty}^{t} \chi(t)\, dN(t) = \chi_0 + \sum_{i=1}^{N(t)} \chi_i, \quad L(x) = \begin{cases} 0, & x \leqq 0 \\ 1, & x > 0 \end{cases},$$

one may define (See Fig. 1)

$$\eta(t) = \int_{-\infty}^{t} \chi(u)\, dN(u) - \int_0^t L(\eta(u))\, du. \tag{1.1}$$

It is sometimes instructive to formally redefine $\chi(t)$ as a stochastic process with $\chi(t)$, $\chi(t')$, $(t \neq t')$, independent, $\Pr\{\chi(t) \leqq x\} = H(x)$, $t > 0$. One then concludes immediately, from the functional form of (1.1) that $\eta(t)$ is a Markov process. Note that var $(\eta(t + \Delta t) - \eta(t)) = O((\Delta t)^2)$, $t_i < t < t_{i+1}$, so that Feller's [5] function $a(t, x) = 0$.

In Section 2, the problem of finding the distribution of $\eta(t)$ will be reduced to finding the unique solution of a Volterra equation of the second kind. In Section 3, the corresponding result is found for the process $\eta^*(t)$, where, if t' is the first zero of $\eta(t)$,

$$\eta^*(t) = \begin{cases} \eta(t), & t < t' \\ 0, & t \geqq t'. \end{cases}$$

The work in Sections 2–4 generalizes results of Beneš [2] who treated the Takács process when $\lambda(t) = \text{const}$ (under somewhat milder restrictions on H). Section 5 contains some results on the asymptotic nature of $\eta(t)$, derived from a more general point of view than that employed in the preceding sections.

2. The Volterra equation for $\Pr\{\eta(t) = 0\}$.

Define $\Lambda(t) = \int_0^t \lambda(u)\, du$, $F(t,x) = \Pr\{\eta(t) \leqq x\}$, $F(t) = F(t,0) = \Pr$

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