## ADMISSIBILITY FOR ESTIMATION WITH QUADRATIC LOSS[1]

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**0. Introduction.** In dealing with estimation of a single unknown parameter, the criteria most commonly employed in evaluating the worth of given estimates is to make comparisons of the expected square deviation of the estimates from the true value. Suppose on the basis of an observation $x$ (or series of observations) on a distribution $P(x, \omega)$ of the form $\int_{-\infty}^{x} p(\xi, \omega)\, d\mu(\xi)$ depending on an unknown parameter $\omega$ it is desired to estimate some function $h(\omega)$. The quantity $p(x, \omega)$ may be regarded as the density of $P(x, \omega)$ with respect to the completely additive measure $\mu$. A non-randomized estimate of $h(\omega)$ is described by a function of the observations $a(x)$, and when the error of an estimate is evaluated in terms of quadratic loss, the expected risk for the estimate $a(x)$ when the true parameter value is $\omega$ is calculated by means of the formula

$$\rho(\omega, a) = \int (a(x) - h(\omega))^2 p(x, \omega)\, d\mu(x). \quad (1)$$

The object is to select the estimate $a$ which minimizes (1) in some sense. The fact that the statistician may restrict attention only to non-randomized estimates is due to the convexity property of the loss function ([1], p. 294; [2], p. 4.3). The justification of the quadratic loss as a measure of the discrepancy of an estimate derives from the following two characteristics: (i) in the case where the $a(x)$ represents an unbiased estimate of $h(\omega)$, (1) may be interpreted as the variance of $a(x)$ and, of course, fluctuation as measured by variance is very traditional in the domain of classical estimation; (ii) from a technical and mathematical viewpoint square error lends itself most easily to manipulation and computations.

Principles used to determine a particular estimate which accomplishes appropriate optimizations are related to the minimax criteria, Bayes procedures, unbiased uniformly minimum variance estimates, etc. However, one prerequisite universally acceptable as desirable for statistical procedures is the property of admissibility. An estimate $a$ is said to be admissible if there exists no other estimate $a^*$ such that $\rho(\omega, a^*) \leqq \rho(\omega, a)$ with inequality for some $\omega$. In other words, an estimating procedure is admissible if it cannot be uniformly improved upon in terms of risk by any other procedure. Certainly, no estimate should be used if we can do better by a different estimate—whatever the true state of nature. It would, therefore, be of considerable interest to establish the admissibility of some of the standard estimates employed in practice.

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