A THREE-SAMPLE KOLMOGOROV-SMIRNOV TEST

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1. Introduction. In 1951, Gnedenko and Korolyuk published an elegant derivation ([6])[1] of the null distribution of the Kolmogorov-Smirnov statistic $D_{2,n}$ for two samples of equal size n. The statistic $D_{2,n}$ is given by

$$D_{2,n} = \sup_t | F_{2,n}(t) - F_{1,n}(t) | , \tag{1}$$

where $F_{i,n}(t)$ is the sample cumulative distribution function for the ith sample. The distribution derived by Gnedenko and Korolyuk is

$$\Pr\left\{ D_{2,n} \geqq \frac{l}{n} \right\} = 2 \binom{2n}{n}^{-1} \sum_{i=1}^{[n/l]} (-1)^{i+1} \binom{2n}{n - il}. \tag{2}$$

Since

$$\lim_{n \to \infty} \frac{\binom{2n}{n - k\sqrt{n}}}{\binom{2n}{n}} = e^{-k^2}, \tag{3}$$

(2) easily leads to the familiar asymptotic result

$$\lim_{n \to \infty} \Pr\left\{ n^{1/2} D_{2,n} \geqq \lambda \right\} = 2 \sum_{i=1}^{\infty} (-1)^{i+1} e^{-(i\lambda)^2}. \tag{4}$$

Gnedenko and Korolyuk's proof hinges on the fact that, in the null case (for two samples drawn from the same continuous distribution), $\Pr\{D_{2,n} \geqq l/n\}$ equals the probability that the maximum deviation from the origin of a certain random walk in the line is at least l. The random paths involved in this random walk start at the origin, and consist of $2n$ unit steps, n to the left and n to the right, with all possible permutations of left and right steps equally likely. The probability $\Pr\{D_{2,n} \geqq l/n\}$ is thus equal to, say, $M \Big/ \binom{2n}{n}$, where $\binom{2n}{n}$ is the total number of equally likely paths, and M is the number of these paths with maximum deviation from the origin at least l. M can be computed by the reflection principle in the line ([2], [1]), leading to (2).

In this paper I show that the null distribution of the three-sample extension $D_{3,n}$ (see (6) below) of $D_{2,n}$ can be derived by extending the geometric approach of [6] from the line to the plane.

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[1] The review of this paper in Mathematical Reviews [3] was brought to my attention by Murray Rosenblatt.