A TABLE FOR COMPUTING TRIVARIATE NORMAL PROBABILITIES

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1. Introduction. For convenience in the following discussion let X, Y, and Z be random variables with a trivariate normal distribution such that $EX = EY = EZ = 0$, $EX^2 = EY^2 = EZ^2 = 1$, $EXY = \rho_{12}$, $EXZ = \rho_{13}$, $EYZ = \rho_{23}$, let $C(h, k, m; \rho_{12}, \rho_{13}, \rho_{23})$ denote the probability that $X \leqq h$, $Y \leqq k$, $Z \leqq m$, and let $D(h, k, m; \rho_{12}, \rho_{13}, \rho_{23})$ denote the probability that $X \geqq h$, $Y \geqq k$, $Z \geqq m$. Several tables have been prepared from which certain particular values of the trivariate normal integral can be obtained. A tabulation of the area of hyperspherical simplices is given by H. Ruben [1]. The function Ruben has tabulated as $\bar{u}_n(x)$ is, for the case $n = 3$, equal to $C(0, 0, 0; 1/x, 1/x, 1/x)$ and the tabulation is for $x = 2(1)11$. This probability can be computed directly, however, as a special case of the well-known formula (for example, see [2]).

$$
\begin{aligned}
C(0, 0, 0; \rho_{12}, \rho_{13}, \rho_{23}) &= D(0, 0, 0; \rho_{12}, \rho_{13}, \rho_{23}) \\
&= \frac{1}{4\pi}(2\pi - \arccos \rho_{12} - \arccos \rho_{13} - \arccos \rho_{23})
\end{aligned} \tag{1.1}
$$

Short tabulations of $C(h, h, h; 1/2, 1/2, 1/2)$ have been published by D. Teichroew [3] for $h\sqrt{2} = 0(.01)6.09$ and by P. N. Somerville [11] for $h = 0(.1)2(.5)3$. In addition to these published tables, there are some unpublished tables [4] giving $C(h, h, h; \rho, \rho, \rho)$ for $\rho = 1/(1 + \sqrt{3})$ and $\frac{1}{4}$, $h = 0(.1)3(.5)8$ and for $\rho = 0(.1)0.9$, $h = 0(.2)1$.

Methods for computing $D(h, k, m; \rho_{12}, \rho_{13}, \rho_{23})$ have been given by M. G. Kendall [5], R. L. Plackett [6], and S. C. Das [7]. The method of Kendall is to express the trivariate normal density as the inverse of its characteristic function obtaining $D(h, k, m; \rho_{12}, \rho_{13}, \rho_{23})$ as a six-dimensional integral. The part of the integral involving the ρ_{ij} is expanded in a power series and the result integrated term by term. The resulting series coverges slowly, however, when the ρ_{ij} are large. Plackett's method, on the other hand, is to consider $D(h, k, m; \rho_{12}, \rho_{13}, \rho_{23})$ as a function of the ρ_{ij} and write it as a line integral from $(\rho_{12}, \rho_{13}, \rho_{23}^*)$ to $(\rho_{12}, \rho_{13}, \rho_{23})$ where ρ_{23}^* is chosen to give a degenerate trivariate normal density so that $D(h, k, m; \rho_{12}, \rho_{13}, \rho_{23}^*)$ becomes a bivariate normal integral. The result of this procedure is that $D(h, k, m; \rho_{12}, \rho_{13}, \rho_{23})$ can be expressed as a sum of lower dimensional normal integrals and an integral which must be evaluated by numerical integration.

The method of Das reduces the trivariate integral to a single integral which is then evaluated numerically provided the correlations are such that their product is positive and each is numerically greater than the product of the other two.

In this paper $C(h, k, m; \rho_{12}, \rho_{13}, \rho_{23})$ is expressed in terms of the univariate

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