ON A PROBABILITY PROBLEM IN THE THEORY OF COUNTERS

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1. Introduction. Let us suppose that particles arrive at a counter in the time interval $(0, \infty)$ according to a Poisson-process of density λ. Each particle arriving in the time interval $(0, \infty)$ independently of the others gives rise to an impulse with probability p or 1 according to whether at this instant there is an impulse present or there is no impulse present. The time durations of the impulses are identically distributed independent positive random variables with distribution function $H(x)$ and these random variables are independent of the instants of the arrivals and of the events of the realizations of the impulses. We define as "registered particles" those particles which occur at an instant when there is no impulse present. Denote by ν_t the number of the registered particles in the time interval $(0, t)$. The problem is to determine the distribution law of ν_t and its asymptotic behaviour as $t \to \infty$.

The particular case of the above problem, when the time durations of the impulses are constant, was investigated earlier by G. E. Albert and L. Nelson [1].

2. The structure of the process. Denote by $\{\tau_n\}$ the sequence of instants at which particles are registered. We say that the system at any instant t is in state A when no impulse covers the instant t and in state B otherwise. Then the system assumes the states A, B, A, B, $\cdots$ alternatingly. Let us denote by ξ_1, η_1, ξ_2, η_2, $\cdots$ the times spent in states A and B respectively. If the system at the instant t is in state A, then t is evidently a regeneration point of the process. Consequently $\{\xi_n\}$ and $\{\eta_n\}$ are independent sequences of identically distributed positive random variables. Clearly $\mathbf{P}\{\xi_n \leqq x\} = F(x) = 1 - e^{-\lambda x}$ if $x \geqq 0$. Write $\mathbf{P}\{\eta_n \leqq x\} = U(x)$, where $U(x)$ is still unknown. (We use $\mathbf{P}$ for the symbol of probability and $\mathbf{E}$ for expectation.) It can easily be seen that the instants of the transitions $A \to B$ coincide with the instants τ_n $(n = 1, 2, \cdots)$. Consequently the time differences $\tau_{n+1} - \tau_n$ $(n = 1, 2, \cdots)$ are identically distributed independent random variables with distribution function $G(x) = F(x) * U(x)$ i.e.

$$G(x) = \int_0^x U(x - y)e^{-\lambda y}\lambda \, dy, \tag{1}$$

while $\mathbf{P}\{\tau_1 \leqq x\} = F(x)$.

3. Notations. Let us introduce the following Laplace-Stieltjes transforms:

$$\gamma(s) = \int_0^\infty e^{-sx} \, dG(x) \tag{2}$$

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