A MARKOVIAN FUNCTION OF A MARKOV CHAIN[1]

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1. Statement of the problem and the results obtained. Consider a Markov chain $X(n), n = 0, 1, 2, \cdots$, with a finite number of states $1, \cdots, m$ and stationary transition probability matrix $P = (p_{ij})$

$$(1) \qquad p_{ij} = P[X(n+1) = j \mid X(n) = i] \geqq 0, \qquad i, j = 1, \cdots, m, \qquad \sum_j p_{ij} = 1.$$

The probability structure of the chain is determined by P and the initial probability distribution vector $p = (p_i)$

$$(2) \qquad p_i = P[X(0) = i] \geqq 0, \qquad i = 1, \cdots, m, \qquad \sum_i p_i = 1.$$

Suppose the experimenter does not observe the process $X(n)$ but rather a derived process $Y(n) = f(X(n))$ where f is a given function on $1, \cdots, m$. The states i of the original process $X(n)$ on which f equals some fixed constant are collapsed into a single state of the new process $Y(n)$. Call these collapsed sets of states $S_i, i = 1, \cdots, r, r \leqq m$. A natural question that arises is as to whether or not the new process is Markovian. It is clear that this is not generally the case.

Let us restrict ourselves to a process $X(n)$ with its initial probability distribution a left invariant vector of the matrix P, that is, $pP = p$. Further assume that all the components of p are positive (all transient states are thrown out). Let D be the diagonal matrix with its ith diagonal entry p_i. The process is said to be reversible if

$$DP = P'D$$

(P' is the transpose of P). The following result is obtained:

THEOREM 1. *Let $X(n)$ be a stationary reversible process with $p_i > 0$ for all i. Then $Y(n)$ is Markovian if and only if for any fixed $\beta = 1, \cdots, r$*

$$(3) \qquad \sum_{j \varepsilon S_\beta} p_{ij} = P[X(n+1) \varepsilon S_\beta \mid X(n) = i] = C_{S_\alpha, S_\beta}$$

has the same value for all i in any given collapsed set of states $S_\alpha, \alpha = 1, \cdots, r$.[2]

A slightly different problem can be phrased in the following way. Let

$$w = (w_i), w_i > 0, i = 1, \cdots, m$$

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[2] J. L. Snell pointed out that the original proof, given for Markov processes $X(n)$ with a symmetric P, holds for the reversible processes.