ON THE KOLMOGOROV AND SMIRNOV LIMIT THEOREMS FOR DISCONTINUOUS DISTRIBUTION FUNCTIONS

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1. Introduction. Let $X_1, X_2, \ldots, X_N$ be N independent random variables with the same distribution function $F(x)$. $S_N(x)$ is the empirical distribution function, i.e., $S_N(x) = k/N$ if exactly k of the N values X_i are less than or equal to x. It is of theoretical and practical interest to analyze the behavior of the statistics

$$\sup_{-\infty<x<\infty} |S_N(x) - F(x)| \cdot N^{\frac{1}{2}}$$

and

$$\sup_{-\infty<x<\infty} (S_N(x) - F(x)) \cdot N^{\frac{1}{2}}.$$

Kolmogorov [12] proved in a famous paper in 1933 that for $\lambda > 0$

$$\text{I} \qquad \lim_{N\to\infty} P[\sup_{-\infty<x<\infty} |S_N(x) - F(x)| \cdot N^{\frac{1}{2}} < \lambda] = \sum_{k=-\infty}^{+\infty} (-1)^k e^{-2\lambda^2k^2}$$

if $F(x)$ is a continuous distribution function. Smirnov [21] obtained a similar result in 1939, when he showed that

$$\text{II} \qquad \lim_{N\to\infty} P[\sup_{-\infty<x<\infty} (S_N(x) - F(x)) \cdot N^{\frac{1}{2}} < \lambda] = 1 - e^{-2\lambda^2}$$

holds for continuous distribution functions $F(x)$.

Kolmogorov converts in his proof to a generalization of the Central Limit theorem, whereas Smirnov's theorem was a corollary to a more intricate theorem. But the two formulae can be proved by reciprocal methods. They have also been proved by Feller [11] and by Doob [10] and Donsker [9]. Feller made use of characteristic functions and Doob employed stochastic processes. Smirnov [22] found in 1944 the first terms of the asymptotic expansion for the probability in II and an exact formula for finite N. Chung [7] and Blackman [5], [6] were successful in finding the asymptotic expansion for the probability in I.

A somewhat more general form of the statistics, namely

$$\sup_{-\infty<x<\infty} |S_N(x) - F(x)| N^{\frac{1}{2}} \cdot \varphi(F(x)),$$

where $\varphi(y)$ is a positive definite weight function, was discussed by Anderson and Darling [1]. They found the limit distributions for some special weight functions,

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