In this note an alternative proof is given which entails little computation and is self-contained.

Replace the interval (0, 1) by the reals modulo 1, considered as a circle of circumference 1. Let c be an arbitrary point on the circle. Moving from c in the direction corresponding to increasing values (0, 1), one meets successively the points $U_{k+1}, U_{k+2}, \cdots, U_n, U_1, \cdots, U_k$ where k, so defined, is a r.v. depending on c. Rename these points $U_1^c, U_2^c, \cdots, U_n^c$ respectively. Define $i = i(j)$ by $U_j^c = U_i$. Let u_j^c denote the (arc) distance of U_j^c from c taken in the increasing direction. Therefore,

$$i = k + j; \qquad u_j^c = U_{k+j} - c \qquad \text{for } j = 1, \cdots, n - k$$

$$i = k + j - n; \qquad u_j^c = U_{k+j-n} + 1 - c \qquad \text{for } j = n - k + 1, \cdots, n$$

With the indicated relation between i and j observe that

$$j/n - u_j^c = (i - k)/n - U_i + c = i/n - U_i + c - k/n.$$

For a fixed c and a given sample, c and k are constants and hence $j/n - u_j^c$ attains its maximum at the same point $U^* = U_{i^*}$ as does $i/n - U_i$.

Given a sample $U_1, \cdots, U_n$, the point U^* on the circle of reals mod. 1 is therefore independent of the choice of the initial point c taken instead of 0 on this circle. Since the distribution of X mod. 1 is uniform, that is, is invariant under translations, the distribution of U^* mod. 1 is also invariant under translations. Thus U^* has a uniform distribution on (0, 1). q.e.d.

REFERENCES

[1] Z. W. Birnbaum and Ronald Pyke, "On some distributions related to the statistic D_n^+," *Ann. Math. Stat.*, Vol. 29 (1958), pp. 179–187.

[2] Meyer Dwass, "On several statistics related to empirical distribution functions," *Ann. Math. Stat.*, Vol. 29 (1958), pp. 188–191.

QUASI-RANGES OF SAMPLES FROM AN EXPONENTIAL POPULATION

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In a study of the use of ranges and quasi-ranges in estimating the standard deviation of a population, Harter [4] has compared the results for samples from a normal population with those for samples from certain other populations, including the exponential. In this note are given the distributions of quasi-ranges from the exponential population and also formulas for the cumulants of these quasi-ranges.

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