SOME BOUNDS ON THE DISTRIBUTION FUNCTIONS OF THE LARGEST AND SMALLEST ROOTS OF NORMAL DETERMINANTAL EQUATIONS[1]

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While the joint density function of the roots of certain determinantal equations have been obtained, [1], [2], [3], the result is sufficiently complex that the marginal distribution functions of these statistics have not, to the author's knowledge, been tabulated. We present here a lower bound on the distribution function of the smallest root and an upper bound on the distribution function of the largest root.

These bounds may be of possible usefulness in problems of significance tests since observed values that are not "significant" according to the bounds will certainly not be "significant" with respect to the exact distribution.

Let S^1_{ij} and S^2_{ij}, $i, j = 1, \cdots, k$, be two sample covariance matrices from normal distributions having identical covariance matrices. It is well known [4] that the smallest and largest roots, say W_1 and W_k, of the equation

$$|S^1_{ij} - WS^2_{ij}| = 0$$

satisfy the inequalities

$$W_1 \leqq \frac{\sum S^1_{ij}\, x_i\, x_j}{\sum S^2_{ij}\, x_i\, x_j} \leqq W_k, \qquad \sum S^2_{ij}\, x_i\, x_j > 0$$

Let $F_i = S^1_{ii} / S^2_{ii}$. It then follows that

$$W_1 \leqq \min_i \{F_i\}; \qquad W_k \geqq \max_i \{F_i\}.$$

Since the roots are invariant under linear transformations of the underlying variables, the covariance matrix may be taken to be the identity matrix. Then the F_i are independently and identically distributed according to the well known F distribution. Denote by $F_{[1]}$ and $F_{[k]}$ the smallest and the largest of a set of k independently identically distributed F values. We then have the desired bounds.

$$P\{W_1 \leqq u\} \geqq P\{F_{[1]} \leqq u\}$$

$$P\{W_k \geqq v\} \geqq P\{F_{[k]} \geqq v\}.$$

Denote by $G(F)$ the distribution function of F (which depends, of course, on the numbers of degrees of freedom for S^1_{ij} and S^2_{ij}). The above bounds become

$$P\{W_1 \leqq u\} \geqq 1 - [1 - G(u)]^k$$

$$P\{W_k \geqq v\} \geqq 1 - [G(v)]^k.$$

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